Mathematical Programs with Equilibrium Constraints: The existence of feasible points.

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Lai-Jiu Lin* and Georg Still†

Abstract

The paper studies three classes of optimization problems with bilevel structure including mathematical programs with equilibrium constraints (MPEC) and semi-infinite problems (SIP). The main goal of the paper is to provide results which establish the existence of feasible points of the problems. These results are based on the so-called KKM Lemma. We are also interested in the convexity properties of these problems.

Keywords: Mathematical problems with equilibrium constraints, semi-infinite problems, bilevel problems, KKM property, properly quasimonotone.

1 Introduction

Generalized semi-infinite problems are programs of the type

$$\text{SIP:} \qquad \min_x \; f(x) \quad \text{s.t.} \quad \phi(x,t) \geq 0 \quad \forall t \in H(x)\ , \tag{1}$$

where $H(x) \subset \mathbb{R}^m$ is the index set defined by a set-valued mapping $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$. Bilevel problems are of the form

$$\begin{array}{lll} \text{BL:} & \min_{x,y} \; f(x,y) \quad \text{s.t.} & g(x,y) \geq 0, \\ & \text{and } y \text{ is a solution of} \quad Q(x): \; \min_t F(x,t) & \text{s.t.} \quad t \in H(x)\ . \end{array} \tag{2}$$

*National Changhua University of Education, Depart. of Mathematics, Changhua, Taiwan, 50058.
†University of Twente, Depart. of Mathematics, 7500AE Enschede, The Netherlands.

We also consider mathematical programs with equilibrium constraints

$$\text{MPEC:} \qquad \min_{x,y} f(x,y) \quad \text{s.t.} \quad \begin{array}{ll} g(x,y) \geq 0\ , \quad y \in H(x) & \\ \text{and} \quad \phi(x,y,t) \geq 0 & \forall t \in H(x)\ . \end{array} \tag{3}$$

These programs represent three important classes of optimization problems which have been investigated in a large number of papers and books (see *e.g.,* [2], [11]-[12] [6] and the references therein). As usual in linear and nonlinear optimization, these papers mainly deal with optimality conditions and numerical methods to solve the problems. Typically the existence of a feasible point is tacitly assumed.

The aim of this paper however is to deal with the latter problem and to investigate under which assumptions the existence of feasible points can be assured in advance. The results are based on the famous KKM Theorem (*Knaster-Kuratowski-Mazurkiewicz*) (see *e.g.* [15]).

The paper is organized as follows. In Section 2, we analyze the relations between the three types of optimization problems, SIP, BL and MPEC. In Section 3, we formulate the existence problems and add some illustrative remarks. In Section 4, based on the KKM approach, we derive results which a-priori establish the existence of feasible points of our problems. Also convexity properties are studied. The Appendix surveys the concepts from convex analysis and functional analysis needed to prove our theorems.

2 Comparison between the structure of the problems

Obviously, SIP is a special case of MPEC. But also BL can be written in MPEC form as follows: If we write the second constraint in BL equivalently as

$$y \in H(x) \quad \text{and} \qquad F(x,t) - F(x,y) \geq 0 \quad \forall t \in H(x)\ .$$

the BL problem becomes

$$\text{BL}_2\text{:} \qquad \min_{x,y} f(x,y) \quad \text{s.t.} \quad \begin{array}{r} g(x,y) \geq 0\ , \quad y \in H(x) \\ \text{and } F(x,t) - F(x,y) \geq 0 \quad \forall t \in H(x)\ . \end{array} \tag{4}$$

Remark. Note however that there is a subtle difference in the interpretation of the constraint

$$\phi(x,t) \geq 0 \ \text{ or } \ \phi(x,y,t) \geq 0 \quad \forall t \in H(x)\ . \tag{5}$$

In the case that $H(x)$ is empty, for BL and MPEC, because of the additional condition $y \in H(x)$, no feasible point (x,y) exists (for this x). For the SIP problem however an empty index set $H(x)$ means that there are no constraints and such points x are feasible.

SIP and MPEC can also be transformed into a problem of bilevel structure. Beginning with the SIP case, we assume $H(x) \neq \emptyset, \ \forall x$, and introduce the (lower level) problem

$$Q(x): \qquad \min_t \ \phi(x,t) \quad \text{s.t.} \quad t \in H(x) \ , \tag{6}$$

depending on the parameter x. Then (assuming that $Q(x)$ is solvable) we can write

$$\phi(x,t) \geq 0 \ \ \forall t \in H(x) \quad \Leftrightarrow \quad \phi(x,y) \geq 0 \text{ and } y \text{ solves } Q(x) \ .$$

So SIP takes the BL form:

$$\begin{array}{lll} \text{SIP}_2\text{:} & \min_{x,y} \ f(x) \quad \text{s.t.} & \phi(x,y) \geq 0, \\ & \text{and } y \text{ is a solution of} \quad Q(x): \ \min_t \phi(x,t) \ \text{ s.t. } \ t \in H(x) \ . & \end{array} \tag{7}$$

This problem is a BL program with the special property that the objective function f does not depend on y and that the constraint function ϕ in the first level coincides with the objective function of the lower level.

To bring MPEC into BL form we apply the same trick as before, and consider the program

$$Q(x,y): \qquad \min_t \ \phi(x,y,t) \quad \text{s.t.} \quad t \in H(x) \ , \tag{8}$$

depending on the parameter (x,y). If we assume that $Q(x,y)$ is solvable, then

$$\phi(x,y,t) \geq 0 \ \ \forall t \in H(x) \quad \Leftrightarrow \quad \phi(x,y,t) \geq 0 \text{ and } t \text{ solves } Q(x,y) \ .$$

Consequently MPEC turns into a problem of BL type (with x replaced by (x,y)):

$$\begin{array}{lll} \text{MPEC}_2\text{:} & \min_{x,y,t} \ f(x,y) \quad \text{s.t.} & g(x,y) \geq 0 \ , \quad y \in H(x), \\ & & \phi(x,y,t) \geq 0 \\ & \text{and } t \text{ is a solution of} \quad Q(x,y): \ \min_u \phi(x,y,u) \ \text{ s.t. } \ u \in H(x) \ . & \end{array} \tag{9}$$

Under the extra condition

$$\phi(x,y,y) = 0 \ , \quad \forall y \ , \tag{10}$$

in MPEC_2 we can 'eliminate' the t variable as follows. In view of (10) we find

$$y \in H(x) \text{ and } \phi(x,y,u) \geq 0 \ \ \forall u \in H(x) \quad \Leftrightarrow \quad y \text{ solves } Q(x,y)$$

and MPEC_2 is equivalent with

$$\begin{array}{lll} \text{MPEC}_3\text{:} & \min_{x,y} \ f(x,y) \quad \text{s.t.} & g(x,y) \geq 0 \ , \quad y \in H(x), \\ & \text{and } y \text{ is a solution of} \quad Q(x,y): \ \min_u \phi(x,y,u) \ \text{ s.t. } \ u \in H(x) \ . & \end{array} \tag{11}$$

Remark. Note however that now the problem MPEC_3 has a more complicated structure than BL, since in MPEC_3 the lower level problem $Q(x,y)$ depends on y which should at the same time solve $Q(x,y)$.

For a comparison from the structural and generic viewpoint between BL and SIP we refer to [12], and between BL and MPEC we refer to [3].

3 The feasibility problems and preliminaries

For the three problems SIP, BL and MPEC we consider the corresponding (basic) feasibility sets:

$$
\begin{aligned}
M_{SIP} &= \{x \in \mathbb{R}^n \mid \phi(x,t) \geq 0 \ \ \forall t \in H(x)\} \\
M^0_{BL} &= \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y \in H(x) \text{ and } F(x,t) - F(x,y) \geq 0 \ \ \forall t \in H(x)\} \\
M^0_{MPEC} &= \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y \in H(x) \text{ and } \phi(x,y,t) \geq 0 \ \ \forall t \in H(x)\}
\end{aligned}
$$

In the next section, conditions are derived which assure that these feasible sets are *non-empty*. Moreover, the structure of the sets is analyzed. Note that if the feasible set is non-empty and compact, the corresponding minimization problems will have a solution provided that the objective function is continuous (lower semicontinuous). We will prove the existence of minimizers without compactness assumptions.

We now add some remarks which could help to illustrate the results of the next section. Let x be fixed and put $K := H(x)$, $\psi(y,t) := \phi(x,y,t)$. The feasible set M^0_{MPEC} then reduces to

$$M = \{y \in \mathbb{R}^m \mid y \in K \text{ and } \psi(y,t) \geq 0 \ \ \forall t \in K \ .\}$$

By defining the (upper level) set $M_{\geq}(y) := \{t \mid \psi(y,t) \geq 0\}$ the feasibility problem $M \neq \emptyset$ can equivalently be expressed in the geometrical form:

$$\text{Find } y \in K \text{ such that } K \subseteq M_{\geq}(y) \ .$$

Considering the parametric program

$$P(y): \qquad v(y) := \min_{t \in K} \ \psi(y,t)$$

this is equivalent to finding an element $y \in K$ such that $v(y) \geq 0$.

Let now K be convex and consider two different cases.

Case that $t \to \psi(y,t)$ is concave: Then the set $M_{\geq}(y)$ is convex and for any y, $P(y)$ is a 'concave' program. (Also $v(y)$ is concave in y.) In particular the minimizer is attained at an extreme point of K.

Case that $t \to \psi(y,t)$ is convex: Then the complement of $M_{\geq}(y)$ is convex and for any y, the problem $P(y)$ is a 'convex' program.

The problems considered in the next section will be of the structure of the second case. The KKM approach in the next section can be seen as a generalization of the following way to prove feasibility based on a *saddle point result.*

Obviously (for compact K) $M \neq \emptyset$ is equivalent with

$$\max_{y \in K} \ \min_{t \in K} \ \psi(y,t) \geq 0 \ .$$

Theorem 1 *Let K be convex and compact. Suppose that $y \to \psi(y,t)$ is quasiconcave and usc (upper semicontinuous) and $t \to \phi(y,t)$ is quasiconvex and lsc (lower semicontinuous). Assume furthermore that for any $t \in K$, there exists an element $y(t) \in K$ such that $\psi(y(t),t) \geq 0$. Then $M \neq \emptyset$.*

Proof. It is well-known (see *e.g.*, [10]) that under the assumptions the relation

$$\max_{y\in K} \min_{t\in K} \psi(y,t) = \min_{t\in K} \max_{y\in K} \psi(y,t) \ .$$

holds. Then by using $\psi(y(t),t) \geq 0$, it follows

$$\max_{y\in K} \min_{t\in K} \psi(y,t) = \min_{t\in K} \max_{y\in K} \psi(y,t) \geq \min_{t\in K} \psi(y(t),t) \geq 0 \ ,$$

which proves the statement. □

We discuss two examples.
Example. Consider the problem:

$$\text{find } y \in K := [-1,1] \text{ such that } \psi(y,t) \geq 0 \ , \quad \forall t \in K \ .$$

for the two cases

$$\psi(y,t) = \psi_1(y,t) := -ty + t^2 \quad \text{and} \quad \psi(y,t) = \psi_2(y,t) := -ty + t^2 + \alpha$$

with given constant $\alpha > 0$. In the first case we have $\psi_1(y,y) = 0$ and $\overline{y} = 0$ is the unique feasible point. In the second case $\psi_2(y,y) = \alpha > 0$ and an easy analysis shows that the feasible set is given by $\mathcal{F}_2 = [-2\sqrt{\alpha}, 2\sqrt{\alpha}]$. Note that for $\alpha < 0$ no feasible point exists.

This is a typical example in the sense that a condition $\psi(y,y) = 0$ will typically lead to (locally) unique solutions whereas a feasible set with interior points occurs in case $\psi(y,y) > 0$ (see also the condition $f(x,y,y) \geq 0$ in Corollary 1).

Example. Given symmetric $m \times m$-matrices A, B, C such that A, C and $C + B - A$ are positive semi-definite and $0 \neq b \in \mathbb{R}^m$, $a \in \mathbb{R}^m$, $\|a\| \leq 1$ ($\|\cdot\|$ the Euclidean norm) we consider the feasibility problem:

$$\text{find } y \in K := \{y \in \mathbb{R}^m \mid \|y\| \leq 1\} \text{ such that } \psi(y,t) \geq 0 \ , \quad \forall t \in K$$

where $\psi(y,t) = -(a^T t)^2 \ (y^T A y) + y^T B t + t^T C t + b^T (y - t)$. By noticing

$$\psi(y,y) = -(a^T y)^2 \ (y^T A y) + y^T B y + y^T C y + b^T (y - y) \geq y^T (C + B - A) y \geq 0$$

we see that the assumptions of Theorem 1 are fulfilled and the above problem has a feasible solution $\overline{y}$. Note that here, $y = 0$ is not feasible.

4 Existence of feasible points and minimizers

In this section, based on the famous KKM Lemma, we will provide theorems establishing the existence of feasible points and the existence of minimizers for the problems above. We do not impose any compactness assumptions. The concepts from convex analysis and functional analysis needed to prove the results are to be found in the Appendix.

We firstly consider the following problem of the semi-infinite type:

$$\min_x \; h(x) \qquad \text{s.t.} \quad x \in M_1 \; , \tag{12}$$

where $M_1 = \{x \in X \mid f(x, y, v) \geq 0 \text{ for all } v \in H(x) \text{ and for all } y \in Y\}$. We prove our basic result.

Theorem 2 *Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be nonempty, convex. Let $f : X \times Y \times Y \to \mathbb{R}$ be continuous, $H : X \rightrightarrows Y$ continuous with nonempty compact values and $T \in KKM(Y, X)$. Suppose that*

(i) *For each compact subset C of Y, $T(C)$ is compact.*

(ii) *For each $(x, v) \in X \times Y$, the function $y \to f(x, y, v)$ is quasiconvex, and*

$$\textit{for any } y \in Y,\, x \in T(y), \textit{ it follows } f(x, y, v) \geqslant 0 \quad \forall v \in H(x) \; . \tag{13}$$

(iii) *There exists a compact subset K of X such that for each finite subset N of Y, there exists a compact convex subset L_N of Y containing N such that for each $x \in T(L_N) \setminus K$, there exists $y \in L_N$ such that $f(x, y, H(x)) \nsubseteq \mathbb{R}_+$;*

Then the set M_1 is nonempty. If in addition H is convex and for each fixed $y \in Y$, and the function $(x, v) \to f(x, y, v)$ is quasiconcave then M_1 is convex.

Suppose that (i),(ii),(iii) holds and M_1 is convex. Suppose further that:

(iv) *h is lsc and quasiconvex;*

(v) *there exists a compact subset D of M_1 such that for each finite subset Q of M_1, there exists a compact convex subset L_Q of M_1 such that for each $x \in L_Q \setminus D$ there exists $y \in L_Q$ such that $h(y) < h(x)$.*

Then there exists a solution of (12).

Proof. (a) Let us define $F(x, y) = f(x, y, H(x))$. Since f is a continuous function and H is a continuous set-valued map with compact values, it follows from Lemma 5 that $F : X \times Y \to \mathbb{R}$ is a continuous set-valued map with nonempty compact values. Define $A : X \rightrightarrows Y$ via

$$A(x) = \{y \in Y \mid F(x, y) \nsubseteq \mathbb{R}_+\}, \text{ where } \mathbb{R}_+ = [0, \infty) \; .$$

We show now that $A(x)$ is convex for each $x \in X$. Indeed, if $y_1,\ y_2 \in A(x),\ \alpha \in [0,1]$, it follows $y_1,\ y_2 \in Y,\ F(x,y_1) \not\subseteq \mathbb{R}_+,\ F(x,y_2) \not\subseteq \mathbb{R}_+$. We want to show that $F(x, \lambda y_1 + (1-\lambda)y_2) \not\subseteq \mathbb{R}_+$ for all $\lambda \in [0,1]$. Suppose that there exists $\lambda_0 \in [0,1]$ such that $F(x, \lambda_0 y_1 + (1-\lambda_0)y_2) \subseteq \mathbb{R}_+$. By quasiconvexity of f wrt. y either

$$F(x,y_1) \subseteq F(x, \lambda_0 y_1 + (1-\lambda_0)y_2) + \mathbb{R}_+ \subseteq \mathbb{R}_+$$

or

$$F(x,y_2) \subseteq F(x, \lambda_0 y_1 + (1-\lambda_0)y_2) + \mathbb{R}_+ \subseteq \mathbb{R}_+ \ .$$

This leads to a contradiction. Therefore, $F(x, \lambda y_1 + (1-\lambda)y_2) \not\subseteq \mathbb{R}_+$ for all $\lambda \in [0,1]$. We also have $\lambda y_1 + (1-\lambda)y_2 \in Y$. Hence $\lambda y_1 + (1-\lambda)y_2 \in A(x)$ and $A(x)$ is convex for all $x \in X$.
Note that for each $y \in Y$, the set $A^-(y)$ is open. Indeed, since $A^-(y) = \{x \in X \mid F(x,y) \cap (-\infty, 0)\} \neq \emptyset$ this follows directly from the fact that F is lsc (see the definition in the appendix with $U = (-\infty, 0)$)

We wish to show that there exists $\bar{x} \in X$ such that $F(\bar{x}, y) \geq 0$ for all $y \in Y$. Suppose that for each $x \in X$, there exist $y \in Y$ and $u \in F(x,y)$ such that $u < 0$. Then $F(x,y) \not\subseteq \mathbb{R}_+$ and $y \in A(x)$. *i.e.,* $A(x) \neq \emptyset$ for all $x \in X$. Then $X = \bigcup\{A^-(y) \mid y \in Y\} = \bigcup\{intA^-(y) \mid y \in Y\}$.

By assumption, for each compact set C of Y, $\overline{T(C)}$ is compact and $L_N \setminus K \subseteq \bigcup\{A^-(y) \mid y \in L_N\} = \bigcup\{\text{int}A^-(y) \mid y \in L_N\}$. So, the assumptions of Lemma 6 hold and there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in T(\bar{y})$ and $\bar{y} \in A(\bar{x})$. Therefore $f(\bar{x}, \bar{y}, H(\bar{x})) \not\subseteq \mathbb{R}_+$ in contradiction to (13). Therefore, there exists $\bar{x} \in X$ such that $F(\bar{x}, y) \geq 0$ for all $y \in Y$ *i.e.,* M_1 is nonempty.

We now show that under the extra conditions the set M_1 is convex. Let $x_1, x_2 \in M_1$, $\lambda \in [0,1]$. Then $f(x_1, y, v) \geq 0$ for all $y \in Y$ and $v \in H(x_1)$, and $f(x_2, y, v) \geq 0$ for all $y \in Y$ and $v \in H(x_2)$. We now prove

$$f(\lambda x_1 + (1-\lambda)x_2, Y, H(\lambda x_1 + (1-\lambda)x_2)) \geq 0 \ .$$

Suppose there exists $\lambda_0 \in [0,1]$ such that $f(\lambda_0 x_1 + (1-\lambda_0)x_2, Y, H(\lambda_0 x_1 + (1-\lambda_0)x_2) \not\subseteq \mathbb{R}_+$. Then there exists $(y_0, v_0) \in Y \times H(\lambda_0 x_1 + (1-\lambda_0)x_2)$ such that $f(\lambda_0 x_1 + (1-\lambda_0)x_2, y_0, v_0) < 0$. Since H is convex, $H(\lambda_0 x_1 + (1-\lambda_0)x_2) \subset \lambda_0 H(x_1) + (1-\lambda_0)H(x_2)$, we find $v_0 = \lambda_0 v_1 + (1-\lambda_0)v_2$ for some $v_1 \in H(x_1)$ and $v_2 \in H(x_2)$. So, $f(\lambda_0 x_1 + (1-\lambda_0)x_2, y_0, \lambda_0 v_1 + (1-\lambda_0)v_2) < 0$. By quasiconcavity of f, either

$$f(x_1, y_0, v_1) \leq \ f(\lambda_0 x_1 + (1-\lambda_0)x_2, y_0, \lambda_0 v_1 + (1-\lambda_0)v_2) < 0$$

or

$$f(x_2, y_0, v_2) \leq \ f(\lambda_0 x_1 + (1-\lambda_0)x_2, y_0, \lambda_0 v_1 + (1-\lambda_0)v_2) < 0 \ ,$$

leading to a contradiction. This show that $\lambda x_1 + (1-\lambda)x_2 \in M_1$ and M_1 is convex.

To prove the existence of a minimizer $\bar{x}$ of (12) let us suppose to the contrary that for each $x \in M_1$, there exists $y \in M_1$ such that $h(y) < h(x)$. Let $S : X \rightrightarrows X$ be defined by $S(x) = \{y \in M_1 \mid h(y) < h(x)\}$. Then $x \notin S(x)$. Since h is lsc, $S^-(y)$ is open for each $y \in X$. Since h is quasiconvex, $S(x)$ is convex. By assumption $L_Q \setminus D \subseteq \bigcup\{S^-(y) \mid y \in L_Q\} = \bigcup\{\mathrm{int}S^-(y) \mid y \in L_Q\}$. Then it follows from Lemma 8 that there exists $\bar{x} \in X$ such that $S(\bar{x}) = \emptyset$. Thus $h(y) \geq h(\bar{x})$ for all $y \in M_1$.

□

We emphasize that under the additional assumptions that X and Y are compact the situation simplifies (see the next remark).

Remark. Suppose that the sets X, Y in Theorem 2 are compact. Then the assumptions (iii), (v) and the quasiconvexity of h are superfluous. It then follows directly that M_1 is nonempty and it is not difficult to show that M_1 is closed (thus compact). So, if h is lsc, a minimizer of (12) exists (without the assumptions iii,iv,v). Note that the condition iii is a sort of a constraint qualification. The same argument applies to all further existence results.

Theorem 2 can be applied to problems of SIP-type in two different ways. Let us firstly consider the SIP program

$$\min_x h(x) \quad \text{s.t.} \quad x \in M_{SIP}$$

where $M_{SIP} = \{x \in X \mid \phi(x, t) \geq 0 \;\; \forall t \in H(x)\}$.

Theorem 3 *Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be nonempty, convex. Let $H : X \rightrightarrows Y$ be a convex continuous map with nonempty compact values and $T \in KKM(Y, X)$. Suppose that*

(i) *for each compact subset C of Y, $T(C)$ is compact and*

$$\textit{for all } y \in Y, x \in T(y), \textit{it follows} \quad \phi(x, v) \geq 0 \quad \forall v \in H(x)\ .$$

(ii) *Suppose that ϕ is affine and $h : X \to \mathbb{R}$ is lsc. and quasiconvex and*

(iii) *there exists a compact subset D of X such that for each finite subset Q of X, there exists a compact convex subset L_Q of X such that for each $x \in L_Q \setminus D$ there exists a $y \in L_Q$ such that $h(y) < h(x)$.*

Then the set M_{SIP} is a nonempty convex set and a solution of SIP exists.

Proof. Let $f : X \times Y \times Y \to \mathbb{R}$ be defined by $f(x, y, v) = \phi(x, v)$, then it is easy to see that f satisfies the conditions of Theorem 2

□

We now consider the special case of SIP:

$$\min_x h(x) \quad \text{s.t.} \quad x \in M'_{SIP}\ , \tag{14}$$

where $M'_{SIP} = \{x \in X \mid \phi(x, y) \geq 0 \;\; \forall y \in Y\}$ with $Y \subset \mathbb{R}^m$. As a corollary of Theorem 2 we now obtain

Theorem 4 *Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be nonempty, convex sets. Let $\phi : X \times Y \to \mathbb{R}$ be continuous, and $T \in KKM(Y, X)$. Let the conditions (iv) and (v) of Theorem 2 hold and suppose that*

(i) *for each compact subset C of Y, $T(C)$ is compact, for each $x \in X$ the function $y \to \phi(x, y)$ is quasiconvex and*

$$\text{for all } y \in Y, x \in T(y), \text{ it follows} \quad \phi(x, y) \geq 0 \ .$$

(ii) *for each $y \in Y$ the function $x \to \phi(x, y)$ is quasiconcave and for each fixed $x \in X$, $y \to \phi(x, y)$ is quasiconvex;*

(iii) *there exist a compact subset K of X such that for each finite subset N of Y, there exists a compact convex subset L_N of Y containing N such that for each $x \in T(L_N) \setminus K$, there exists $y \in L_N$ such that $\phi(x, y) < 0$.*

Then the set M'_{SIP} is nonempty and convex, and there exists a solution of the SIP in (14).

Proof. Let $f : X \times Y \times Y \to \mathbb{R}$ be defined by $f(x, y, v) = \phi(x, y)$, then it is easy to see that f satisfies the conditions of Theorem 2.

□

We emphasize that for the special case $Y = X$ in the preceding Theorem, the condition for T is superfluous since it is trivially satisfied by the mapping $T(x) = \{x\}$.

We now come to the problems of MPEC type and define the mapping $M : X \rightrightarrows Y$ by

$$M(x) = \{y \in H(x) \mid f(x, y, v) \geq 0 \text{ for all } v \in H(x)\} \ . \tag{15}$$

Definition 1. A function $f : \mathbb{R}^{2m} \to \mathbb{R}$ is called properly quasimonotone on the convex set $Y \subseteq \mathbb{R}^m$ if for every finite set $\{v_1, v_2, \ldots, v_n\} \subseteq Y$ the following holds:

$$\inf_{y \in co\{v_1, v_2, \ldots, v_n\}} \max_{1 \leqslant j \leqslant k} f(y, v_j) \geq 0$$

Lemma 1

(a) *Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be nonempty, and convex. Let $f : X \times Y \times Y \to \mathbb{R}$ be usc and let $H : X \rightrightarrows Y$ be a usc multivalued map with nonempty, closed and convex values. Suppose*

(i) *the function $f_x : \mathbb{R}^{2m} \to \mathbb{R}$, $f_x(y, v) := f(x, y, v)$ is properly quasimonotone;*

(ii) *for each $x \in X$ there exists a nonempty compact subset K_x of Y such that for each finite subset N_x of $H(x)$, there exist a compact convex subset L_{N_x} of $H(x)$ containing N_x such that for each $y \in L_{N_x} \setminus K_x$ there exists $v' \in L_{N_x}$ such that $y \in H(x)$ and $f(x, y, v') < 0$.*

Then for each $x \in X$ the set $M(x)$ is a nonempty, closed subset of Y.

(b) *If moreover $y \to f(x, y, v)$ is quasiconcave for every $x \in X$, $v \in Y$, the set $M(x)$ (*cf. *(15) is also convex valued.*

Proof. (a) For each $x \in X$, let $Q_x : H(x) \rightrightarrows H(x)$ be defined by

$$Q_x(v) = \{y \in H(x) \mid f(x, y, v) \geq 0\} .$$

We wish to show that Q_x is a KKM map. Suppose it is not. Then there exists a finite set $\{v_1, ..., v_n\}$ in $H(x)$ such that $co\{v_1, ...v_n\} \nsubseteq \cup_{i=1}^n Q_x(v_i)$. Therefore there exists $y \in co\{v_1, ...v_n\}$ such that $y \notin Q_x(v_i)$ for all $i = 1, ..., n$. Since $H(x)$ is convex and $v_1, ..., v_n \in H(x)$, we find $y \in co\{v_1, ...v_n\} \subset H(x)$. By $y \notin Q_x(v_i)$, we have $f(x, y, v_i) < 0$ for all $i = 1, ..., n$. Since f_x is properly quasimonotone there exists $1 \leq j \leq n$ such that $f(x, y, v_j) \geq 0$. This leads to a contradiction showing that Q_x is a KKM map for each $x \in X$.

Since f is usc, for each $x \in X$ and $v \in Y$, the set $Q_x(v)$ is closed. Since for each $x \in X$, there exists a nonempty compact subset K_x of $H(x)$ such that for each finite subset N_x of $H(x)$, there exist a compact convex subset L_{N_x} of $H(x)$ containing N_x such that for each $x \in X$, $L_{N_x} \setminus K_x \subseteq \bigcup\{(Q_x(v))^c \mid v \in L_N\}$. From Lemma 4, we deduce that $\bigcap\{Q_x(v) \mid v \in H(x)\} \neq \emptyset$. Therefore for each $x \in X$, there exists $y \in H(x)$ such that $f(x, y, v) \geq 0$ for all $v \in H(x)$. This shows that $M(x) \neq \emptyset$ for each $x \in X$.

For each $x \in X$, $M(x)$ is closed. Indeed, if $y \in \overline{M(x)}$, then there exists a sequence $y_n \in M(x)$ such that $y_n \to y$. One has $y_n \in H(x)$, $f(x, y_n, v) \geq 0$ for all $v \in H(x)$. Since H is an usc set-valued map with closed values, it follows from Lemma 3 that H is closed. Therefore $y \in H(x)$. We have $f(x, y_n, v) \geq 0$. Since f is usc, this implies $f(x, y, v) \geq \overline{lim} f(x_n, y_n, v) \geq 0$. Consequently, $y \in M(x)$ and $M(x)$ is closed.
(b) We finally show that $M(x)$ is convex for each $x \in X$ if the quasiconcavity condition for f is satisfied. Indeed, if $y_1, \ y_2 \in M(x)$ and $\lambda \in [0, 1]$ it follows $f(x, y_1, v) \geq 0$, $f(x, y_2, v) \geq 0$ for all $v \in H(x)$ and $y_1, \ y_2 \in H(x)$. We have $y_\lambda = \lambda y_1 + (1 - \lambda)y_2 \in H(x)$. Since $f(x, y, v)$ is quasiconcave wrt. y for any $v \in H(x)$, either

$$f(x, y_\lambda, v) \geq f(x, y_1, v) \geq 0 \quad \text{or} \quad f(x, y_\lambda, v) \geq f(x, y_2, v) \geq 0$$

Therefore $y_\lambda \in M(x)$ and $M(x)$ is convex.

□

Remark. Condition (ii) of Lemma 1 (a) is satisfied if $H : X \rightrightarrows Y$ is a multi-valued map with nonempty compact values.

The next lemma provides us with a sufficient condition for proper quasimonotonicity.

Lemma 2 *Let $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ be given and define $f_x(y, v) = f(x, y, v)$. Suppose that for each fixed $x \in X$, the relation $f(x, y, y) \geq 0$ holds for all $y \in Y$ and for each fixed $(x, y) \in X \times Y$, the mapping $v \to f(x, y, v)$ is quasiconvex. Then $f_x : \mathbb{R}^{2m} \to \mathbb{R}$ is properly quasimonotone.*

Proof. Since for each fixed $(x, y) \in X \times Y$, the function $v \to f(x, y, v)$ is quasiconvex, we see that for any set $\{v_1, v_2, \dots, v_n\}$ in Y, and any $y \in co\{v_1, v_2, \dots, v_n\}$, there exists $1 \le j \le n$ such that $f(x, y, v_j) \ge f(x, y, y) \ge 0$. Therefore

$$\inf_{y \in co\{v_1, v_2, \dots, v_n\}} \max_{1 \le i \le n} f(x, y, v_i) \ge 0 \quad \text{or} \quad \inf_{y \in co\{v_1, v_2, \dots, v_n\}} \max_{1 \le i \le n} f_x(y, v_i) \ge 0$$

showing that f_x is properly quasimonotone.

□

Lemma 2 immediately leads to

Corollary 1 *If in Lemma 1 the assumption that $f_x(y, v) := f(x, y, v)$ is proper quasimonotone is replaced by: for any $(x, y) \in X \times Y$,*

$$f(x, y, y) \ge 0 \quad \textit{and} \quad v \to f(x, y, v) \textit{ is quasiconvex },$$

then the same conclusion holds.

The result of Lemma 1 will now lead to existence results for the minimization problem

$$\min_{(x,y)} \; h(x, y) \quad \text{s.t.} \quad (x, y) \in M^0_{MPEC} \, , \tag{16}$$

where $M^0_{MPEC} = \{(x, y) \in X \times Y \mid y \in H(x) \text{ and } f(x, y, t) \ge 0 \;\; \forall t \in H(x)\}$.

Theorem 5
(a) *If f is quasiconcave and H is convex and concave, then the set M^0_{MPEC} is convex (see also [3]).*
(b) *Let the assumptions of Lemma 1(a) hold. Suppose further that*

(i) *the function h is lsc and quasiconvex;*
(ii) *there exists a compact subset D of M^0_{MPEC} , such that for each finite subset Q of M^0_{MPEC}, there exists a compact convex subset L_Q containing Q such that for each $(x, y) \in L_Q \setminus D$, there exist $(u, v) \in L_Q$ such that $h(u, v) < h(x, y)$.*

Then the set M^0_{MPEC} is nonempty and there exists a minimizer of (16).

Proof. (a) To derive the convexity result for M^0_{MPEC} consider (x_1, y_1), $(x_2, y_2) \in M^0_{MPEC}$ and $\lambda \in [0, 1]$. Then $(x_i, y_i) \in X \times Y$, $y_i \in H(x_i)$ and $f(x_i, y_i, v) \ge 0$ for all $v \in H(x_i)$, $i = 1, 2$. We have $(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) \in X \times Y$. Since H is concave, $\lambda y_1 + (1-\lambda)y_2 \in H(\lambda x_1 + (1-\lambda)x_2)$. Now choose any $v \in H(\lambda x_1 + (1-\lambda)x_2)$. Since H is convex, *i.e.*, $v \in H(\lambda x_1 + (1-\lambda)x_2) \subseteq \lambda H(x_1) + (1-\lambda)H(x_2)$, *i.e.*, there exists $v_1 \in H(x_1)$, $v_2 \in H(x_2)$ such that $v = \lambda v_1 + (1-\lambda)v_2$. By quasiconcavity of f, either

$$0 \leq f(x_1, y_1, v_1) \leq f(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2, \lambda v_1 + (1-\lambda)v_2) \text{ or}$$

$$0 \leq f(x_2, y_2, v_2) \leq f(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2, \lambda v_1 + (1-\lambda)v_2)$$

In any case $f(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2, v) \geq 0$, implying $\lambda(x_1, y_1) + (1-\lambda)(x_2, y_2) \in M^0_{MPEC}$. So M^0_{MPEC} is a convex subset of $X \times Y$.

(b) By Lemma 1 for any $x \in X$ the set $M(x)$ is nonempty. So, in particular, $M^0_{MPEC} \neq \emptyset$. To prove that a minimizer of (19) exists let us suppose that there is no minimizer. Then for all $(x, y) \in M^0_{MPEC}$, there exists $(u, v) \in M^0_{MPEC}$ such that $h(u, v) < h(x, y)$. Let the mapping $S : M^0_{MPEC} \multimap M^0_{MPEC}$ be defined by $S(x, y) = \{(u, v) \in M^0_{MPEC} |\ h(u, v) - h(x, y) < 0\}$. Then $S(x, y) \neq \emptyset$ for all $(x, y) \in X \times Y$. Clearly, $(x, y) \notin S(x, y)$. Since h is lsc, $S^-(u, v)$ is open for each $(u, v) \in M^0_{MPEC}$. Since h is quasiconvex, the set $S(x, y)$ is convex for each $(x, y) \in M^0_{MPEC}$. By assumption we find

$$L_Q \setminus D \subseteq \bigcup \{S^-(u, v) \mid (u, v) \in M^0_{MPEC}\} = \bigcup \{\, int S^-(u, v) \mid (u, v) \in M^0_{MPEC}\}\ .$$

Then it follows from Lemma 8 that there exists $(\bar{x}, \bar{y}) \in M^0_{MPEC}$ such that $S(\bar{x}, \bar{y}) = \emptyset$. Therefore, $f(u, v) \geq f(\bar{x}, \bar{y})$ for all $(u, v) \in M^0_{MPEC}$. This leads to contradiction and a minimizer of (19) exists.

□

We now derive the results for the problems

$$\min_{(x,y)} \ h(x, y) \quad \text{s.t.} \quad (x, y) \in M_{MPEC}\ , \tag{17}$$

$M_{MPEC} = \{(x, y) \in X \times Y \mid g(x, y) \geq 0,\ y \in H(x) \text{ and } f(x, y, t) \geq 0\ \ \forall t \in H(x)\}$.

Theorem 6 *In addition to the assumptions in Theorem 5(b), let $g : X \times Y \to \mathbb{R}$ be quasiconcave. Suppose that $H : X \rightrightarrows Y$ maps according to $H|_A : A \rightrightarrows B$ and graph$H|_A \subseteq \mathcal{F}$, where $A = \pi_X \mathcal{F}$, $B = \pi_Y \mathcal{F}$, $\mathcal{F} = \{(x, y) \in X \times Y \mid g(x, y) \geq 0\}$,* i.e., *$A$ is the standard projection of $\mathcal{F}$ on X and B is the standard projection of $\mathcal{F}$ on Y. Suppose furthermore that $\mathcal{F}$ is nonempty. Then the set M_{MPEC} is nonempty and a minimizer of (17) exists.*

Proof. It is easy to see that $\mathcal{F}$ is a nonempty, convex subset of $X \times Y$. Then also A is a convex subset of $X \subset \mathbb{R}^n$ and B a convex subset of $Y \subset \mathbb{R}^m$ and $H|_A : A \rightrightarrows B$. By Theorem 5(b) there exists $(\bar{x}, \bar{y}) \in A \times B$ such that $\bar{y} \in H(\bar{x})$ and $f(\bar{x}, \bar{y}, v) \geq 0$ for all $v \in H(\bar{x})$. Then $(\bar{x}, \bar{y}) \in$ graph$H|_A \subseteq \mathcal{F}$. Hence $g(\bar{x}, \bar{y}) \geq 0$. This shows that M_{MPEC} is nonempty.

□

Corollary 2 *Let $X,\ Y,\ g,\ h,\ H$ be as in Theorem 6 and let the assumptions of Theorem 5(b) hold. Let $F : X \times Y \to \mathbb{R}^m$ be a continuous function and $p : X \to \mathbb{R}$ be a continuous convex function. Then there exists a solution $(\bar{x}, \bar{y}) \in X \times Y$ of the problem:*

$$\min_{(x,y)} \ h(x, y) \qquad \text{s.t. } (x, y) \in X \times Y,\ g(x, y) \geq 0,\ y \in H(x),$$
$$\text{and} \quad \langle F(x, y), y - v\rangle + p(v) - p(y) \geq 0 \ \text{ for all } \ v \in H(x).$$

Proof. The function $f(x, y, v) := \langle F(x, y), y - v\rangle + p(v) - p(y)$ satisfies $f(x, y, y) = 0$ and for each $(x, y) \in X \times Y$ the function $v \to f(x, y, v)$ is quasiconvex. f is a continuous function. By Lemma 2 the function $f_x : Y \times Y \to \mathbb{R}$ is properly quasimonotone. So the result follows from Theorem 6.

□

Remark .
(1) The results of Theorems 5 and 6 remain true if the assumption that f_x is properly quasimonotone is replaced by the sufficient condition of Lemma 2.
(2) Corollary 2 is an existence theorem for mathematical programs with mixed variational inequality constraints. This type of program contains mathematical programs with variational inequality constraints and mathematical bilevel programs.

We now can establish existence results for bilevel problems

$$\min_{(x,y)} \ h(x, y) \quad \text{s.t.} \quad (x, y) \in M_{BL}\ ,$$

where $M_{BL} = \{(x, y) \in X \times Y \mid g(x, y) \geq 0,\ y \in H(x) \text{ and } y \text{ solves } \min_{v \in H(x)} \Psi(x, v)\ \}$

Corollary 3 *Let $X,\ Y,\ h,\ H$ and g be as in Theorem 6. Let $\Psi : X \times Y \to \mathbb{R}$ be an affine function. Then the set M_{BL} is nonempty.*

Proof. Consider the form (4) of a BL and define $f(x, y, v) := \Psi(x, v) - \Psi(x, y)$. Then f is a quasiconcave and continuous function, $f(x, y, y) = 0$ and for fixed $(x, y) \in X \times Y$ the function $v \to f(x, y, v)$ is quasiconvex for each $(x, y) \in X \times Y$. Therefore f_x is properly quasimonotone. So, the conclusion follows directly from Theorem 6.

□

As a simple consequences of Theorem 6 we obtain another existence result for problems of semi-infinite type.

Theorem 7 *Let $X,\ Y$, and g be as in Theorem 6. Let $f : X \times Y \to \mathbb{R}$ be a continuous function. Suppose the mapping f is affine, $H : X \rightrightarrows Y$ maps according to $H|_A : A \rightrightarrows B$, and $\text{graph} H|_A \subseteq \mathcal{F}$, where $A = \pi_x \mathcal{F}$, $B = \pi_y \mathcal{F}$ with $\mathcal{F} = \{(x, y) \in X \times Y \mid g(x, y) \geq 0,\ f(x, y) \geq 0\}$. Suppose, $\mathcal{F}$ is nonempty.*
Suppose h is lsc and quasiconvex and there exists a compact subset K of $\mathcal{F}$ such that

for each finite subset Q of $\mathcal{F}$, there exists a compact convex subset L_Q of $\mathcal{F}$ containing Q such that for each $(x, y) \in L_Q \setminus K$, there exists $(u, v) \in L_Q$ such that $h(u) < h(x)$. Then there exists a solution of the program:

$$\min_{x,y} \; h(x) \qquad s.t. \quad (x, y) \in X \times Y, \;\; g(x, y) \geq 0, \;\; y \in H(x)$$
$$\text{and } f(x, v) \geq 0 \text{ for all } v \in H(x).$$

Proof. It is easy to see that $\mathcal{F}$ is a nonempty convex subset of $X \times Y$. A is a nonempty convex subset of X and B is a nonempty convex subset of Y. Let $q(x, y, v) = f(x, v) - f(x, y)$. Then the mapping q is an usc function, $q(x, y, y) = 0$ and the mapping $(x, y, v) \to q(x, y, v)$ is quasiconvex. Therefore $q_x(y, v) = q(x, y, v)$ is properly quasimonotone. Then it follows from Theorem 5 that there exists (x, y) such that $(x, y) \in \mathcal{F}$, $y \in H(x)$ and $q(x, y, v) = f(x, v) - f(x, y) \geq 0$ or $f(x, v) \geq f(x, y) \geq 0$ for all $v \in H(x)$. So the feasible set of the problem above is nonempty. Let us define $M''_{SIP} = \{x \in X | \; g(x, y) \geq 0$ for some $y \in H(x)$ and $f(x, v) \geq 0$ for all $v \in H(x)\}$. By Theorem 5 M''_{SIP} is a nonempty convex subset of X. By following the same arguments as in the proof of Theorem 5 we obtain the result.

□

5 Appendix

In this section, we survey the definitions, concepts and results needed to prove the main theorems in the preceding section.

Let $T : X \rightrightarrows Y$ be a set-valued map from a space X to another space Y. We denote $\text{graph}(T) = \{(x, y) \in X \times Y \mid y \in T(x)\}$ the graph of T. The inverse T^- of T is the set-valued map $T^- : T(X) \rightrightarrows X$ defined by $x \in T^-(y)$ if and only if $y \in T(x)$.

Let X and Y be topological spaces (abbreviated by ts). A set-valued map $T : X \rightrightarrows Y$ is called *closed* if $\text{graph}(T)$ is a closed subset of $X \times Y$. T is said to be upper semicontinuous (in short usc) (respectively lower semicontinuous (in short lsc)) at $x \in X$, if for every open set U in Y with $T(x) \subseteq U$ (resp. $T(x) \cap U \neq \emptyset$) there exists an open neighborhood $V(x)$ of x such that $T(x') \subseteq U$ (resp. $T(x') \cap U \neq \emptyset$) for all $x' \in V(x)$. T is said to be usc (resp. lsc) on X if T is usc (resp. lsc) at every point of X. T is continuous at x if T is both usc and lsc at x. Note that by definition (*cf.* also [13]) $T : X \rightrightarrows Y$ is lsc at $x \in X$ if and only if for any $y \in T(x)$ and any net $\{x_\alpha\}$ in X converging to x, there is a net $\{y_\alpha\}$, $y_\alpha \in T(x_\alpha)$ with y_α converging to y.

Let X be a convex subset of a topological vector space (in short tvs) and let Y be a ts. A set-valued mapping $T : X \rightrightarrows X$ is called a KKM mapping if for any finite subset N of X:

$$(\text{co } N) \subseteq T(N) = \bigcup \{T(x) \mid x \in N\} \; .$$

Consider now set-valued mappings $S, T : X \rightrightarrows Y$. Then S is said to be a generalized

KKM mapping w.r.t. T if for each finite subset N of X the following holds

$$T(\text{co } N) \subseteq S(N) = \bigcup \{S(x) \mid x \in N\} .$$

Here co N denotes the convex hull of N. The set-valued map $T : X \rightrightarrows Y$ is said to have the KKM property (*cf.* [4]) if $S : X \rightrightarrows Y$ is a generalized KKM mapping w.r.t. to T such that the family $\{\overline{S(x)} \mid x \in X\}$ has the finite intersection property (*i.e.*, any finite subcollection has a nonempty intersection). We denote by $KKM(X, Y)$ the family of all set-valued maps having the KKM property.

Lemma 3 *(*cf. *[1]) Let X, Y be two Hausdorff tvs and $T : X \rightrightarrows Y$ be a set-valued map:*

(i) If X is compact and T is usc with compact values, then $T(X)$ is compact.

(ii) If Y is compact and T is closed, then T is closed valued.

(iii) If T is an usc set-valued map with closed values, then T is closed.

Lemma 4 *(*cf. *[15]) Let E be a Hausdorff tvs, Y be a convex subset of E, X be a nonempty subset of Y, $F : X \rightrightarrows Y$ be a set-valued map satisfying the following conditions:*

(i) for each finite subset $\{x_1, \ldots, x_n\}$ of X, co $\{x_1, \ldots, x_n\} \subset \bigcup_{i=1}^{n} F(x_i)$.

(ii) for all $x \in X$, $F(x)$ is closed and there exists $x_0 \in X$ such that $F(x_0)$ is compact.

(iii) There exists a nonempty compact subset K of Y such that for each finite subset N of Y, there exists a compact convex subset L_N of Y containing N such that $L_N \cap \{\cap F(x) : x \in L_N\} \subseteq K$.

Then $\bigcap_{x \in X} F(x) \neq \emptyset$.

Lemma 5 *(*cf. *[7]) Let E_1, E_2 and Z be Hausdorff ts, X and Y be nonempty subsets of E_1 and E_2 respectively, $F : X \times Y \times X \rightrightarrows Z$ and $S : X \rightrightarrows X$*

(a) If S and F are both lsc, then $T : X \times Y \rightrightarrows Z$ defined by $T(x, y) = \bigcup_{u \in S(x)} F(x, y, u) = F(x, y, S(x))$ is lsc on $X \times Y$.

(b) If S and F are both usc with compact values, then T is an usc set-valued map with compact values.

Lemma 6 *(cf. [8]) Let Y be a convex Hausdorff tvs, let X be a Hausdorff ts and let $T \in KKM(Y, X)$. Suppose that for each compact subset C of Y, $T(C)$ is compact. Let $P : X \rightrightarrows Y$ be a set-valued map such that for all $x \in X$, $P(x)$ is convex and $X = \bigcup\{int \ \ P^{-}(y) \mid y \in Y\}$.*
Suppose that there exists a compact subset K of X such that for each finite subset N of Y, there exists a compact convex subset L_N of Y containing N such that $T(L_N) \setminus K \subseteq \bigcup\{int P^{-}(y) \mid y \in L_N\}$. Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in T(\bar{y})$ and $\bar{y} \in P(\bar{x})$.

Lemma 7 *([14]) Let X be a nonempty convex subset of a Hausdorff ts E , $g : X \times X \to \mathbb{R}$ and $f : X \times X \to \mathbb{R}$ suppose the following conditions are satisfied:*

(i) *for each $x \in X$, $g(x, x) \geq 0$;*
(ii) *for each $y \in X$ the function $f(\cdot, y)$ is usc*
(iii) *for all $x, y \in X$, $f(x, y) < 0$ implies $g(x, y) < 0$;*
(iv) *for all $x, y \in X$ the function $g(x, \cdot) : X \to \mathbb{R}$ is quasiconvex;*
(v) *there exists a nonempty compact set $A \subseteq E$ and a compact convex subset $B \subseteq X$ such that for each $x \in X \setminus A$ there exists $y \in B$ such that $f(x, y) < 0$*

Then there exists $\bar{x} \in A$ such that $f(\bar{x}, y) \geq 0$ for all $y \in X$.

As usual a function f is called quasiconvex if all lower level sets are convex. and quasiconcave if $-f$ is quasiconvex.
Let X, Y be vector spaces. A set-valued mapping $H : X \rightrightarrows Y$ is called convex if for any $x_1, x_2 \in X$ and $0 \leq \lambda \leq 1$ the relation

$$H(\lambda x_1 + (1-\lambda)x_2) \subseteq \lambda H(x_1) + (1-\lambda)H(x_2)$$

is satisfied and concave if the inverse inclusion $\supseteq$ holds.

The following Lemma is a special case of Theorem 7 in [5].

Lemma 8. Let X be a nonempty convex subset of a tvs E. Let $S, T : X \rightrightarrows X$ be multivalued maps satisfying the following conditions :

(i) for all $x \in X$, $coS(x) \subseteq T(x)$;

(ii) for all $x \in X$, $x \notin T(x)$ and $S^{-}(y)$ is open for each $y \in X$;

(iii) there exists a nonempty compact convex subset $C \subseteq X$ and a nonempty compact subset K of X such that for each $x \in X \setminus K$, there exists $\bar{y} \in C$ such that $x \in S^{-}(\bar{y})$.

Then there exists $\bar{x} \in X$ such that $S(\bar{x}) = \emptyset$.

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