EQUILIBRIUM CONSTRAINED OPTIMIZATION PROBLEMS

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ABSTRACT. We consider equilibrium constrained optimization problems, which have a general formulation that encompasses well-known models such as mathematical programs with equilibrium constraints, bilevel programs, and generalized semi-infinite programming problems. Based on the celebrated KKM lemma, we prove the existence of feasible points for the equilibrium constraints. Moreover, we analyze the topological and analytical structure of the feasible set. Alternative formulations of an equilibrium constrained optimization problem (ECOP) that are suitable for numerical purposes are also given. As an important first step for developing efficient algorithms, we provide a genericity analysis for the feasible set of a particular ECOP, for which all the functions are assumed to be linear.

KEYWORDS. Equilibrium problems, existence of feasible points, mathematical programs with equilibrium constraints, problems with complementarity constraints, bilevel programs, generalized semi-infinite programming, genericity.

1. INTRODUCTION

An *equilibrium constrained optimization problem* (ECOP) is a mathematical program, for which an embedded set of constraints is used to model the equilibrium conditions in various applications. This *equilibrium* concept corresponds to a desired state such as the optimality conditions for the inner problem of a bilevel optimization model, or the Nash equilibrium of a game played by rational players. For an introduction to ECOP we refer to [14] and [15]. Applications of ECOP appear not only in economics (Cournot oligopoly, Stackelberg games, generalized Nash equilibrium) but also in optimum design problems in mechanics (contact problems with friction, elasticity problems with obstacles etc., see [15]).

This paper is concerned with the analysis of some structural properties of an ECOP. In order to pursue this analysis, we frequently use standard terms from generalized convexity and set valued analysis. For an unfamiliar reader, we have added an appendix section (Appendix A) that reviews the definitions of these terms.

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Let $f : \mathbb{R}^{n+m} \to \mathbb{R}$, $\phi : \mathbb{R}^{n+2m} \to \mathbb{R}$ be real valued functions and $K : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ a set valued mapping with closed values. A general form of an ECOP is now given by

$$\begin{array}{ll} \min\limits_{x,y} & f(x,y) \\ \text{s.t} & (x,y) \in Z \\ & y \in K(x) \\ & \phi(x,y,v) \geq 0, \ \ \forall v \in K(x)\ , \end{array} \tag{1.1}$$

where $x \in \mathbb{R}^n, y, v \in \mathbb{R}^m$ and the set $Z \subseteq \mathbb{R}^{n+m}$ is a closed nonempty set. The constraints

$$\phi(x,y,v) \geq 0, \ \ \forall v \in K(x), \tag{1.2}$$

depending on the parameter x and y, are called the parametric equilibrium constraints. For notational convenience, we now introduce the so-called graph(K) (see [2]) of the set valued mapping K given by

$$\text{graph}(K) := \{(x,y) \in \mathbb{R}^{n+m} : y \in K(x)\}$$

and the set $E \subseteq \mathbb{R}^{n+m}$ defined by

$$E := \{(x,y) \in \mathbb{R}^{n+m} : \phi(x,y,v) \geq 0, \ \ \forall v \in K(x)\}.$$

This notation allows us to denote the feasible set of (1.1) by

$$\mathcal{F} := Z \cap E \cap \text{graph}(K). \tag{1.3}$$

Hence, we can rewrite the ECOP as follows.

$$\begin{array}{ll} \min\limits_{x,y} & f(x,y) \\ \text{s.t} & (x,y) \in \mathcal{F}. \end{array} \tag{1.4}$$

A frequently used instance of (1.2) arises when for every x the set $K(x)$ is closed, convex, and the function ϕ is given by

$$\phi(x,y,v) := \langle v - y, F(x,y) \rangle. \tag{1.5}$$

The parametric equilibrium constraints (1.2) associated with the function ϕ in (1.5) and the closed convex set $K(x)$, are called the (parametric) Stampacchia variational inequalities. Moreover, it is well-known (see [9]) that if the function $y \to F(x,y)$ in (1.5) is pseudomonotone (see Definition A.1), then the function ϕ can be replaced by

$$\phi(x,y,v) := \langle v - y, F(x,v) \rangle. \tag{1.6}$$

Accordingly, the parametric equilibrium constraints defined by the function ϕ in (1.6) are known as the (parametric) Minty variational inequalities. Notice that in the literature an ECOP is called a mathematical program with equilibrium constraints (MPEC) when ϕ has the form (1.5). In this paper we have chosen the more general form (1.2) so that in addition to MPECs, our model also includes bilevel programs and semi-infinite problems.

In Section 2 of this paper we investigate under which sufficient conditions on the set valued mapping K and the function ϕ, the set $E \cap \text{graph}(K)$ is nonempty. In Section 3 we then study under which conditions on K and ϕ, the set $E \cap \text{graph}(K)$ is closed and convex. In Section 4 we derive different formulations of an ECOP as a nonlinear programming problem. We are especially interested in formulations, which are suitable for numerical purposes. Finally, in Section 5 we give a genericity analysis for the structure of the feasible set of a linear ECOP (where all the problem functions are linear). This genericity analysis constitutes the first step towards developing efficient algorithms.

2. Existence of Feasible Solutions

In this section we are interested in some sufficient conditions, which guarantee that the equilibrium constraints defining the set $E \cap \text{graph}(K)$, allow feasible points. By the definition of the sets E and graph(K), it is clear that $E \cap \text{graph}(K) \neq \emptyset$ if and only if there exists some $x \in \mathbb{R}^n$ such that

$$V(\phi, K(x)) := \{y \in K(x) : \phi(x, y, v) \geq 0, \;\; \forall v \in K(x)\} \neq \emptyset.$$

From now on, we fix x arbitrarily, define $C \in \Re^m$ by $C := K(x)$ and $\phi_x : \mathbb{R}^{2m} \to \mathbb{R}$ by $\phi_x(y, v) := \phi(x, y, v)$, and assume that C is nonempty and convex. Recall that by our general assumption in Section 1, the set C is also closed. First observe that

$$V(\phi_x, C) = \cap_{v \in C} \Phi(v) \;, \tag{2.1}$$

where the set valued mapping $\Phi : C \rightrightarrows C$ is defined by

$$\Phi(v) := \{y \in C : \phi_x(y, v) \geq 0\}. \tag{2.2}$$

In order to prove that the set $V(\phi_x, C)$ is nonempty, we will apply to relation (2.1) the celebrated lemma of *Knaster-Kuratowski-Mazurkiewicz* (KKM lemma) discussed in the Appendix. If we additionally know that the set $\Phi(v)$ is convex for every $v \in C$ (this holds if the function $y \to \phi_x(y, v)$ is quasiconcave (see Definition A.2) for every $v \in C$), then the KKM lemma is a direct consequence of the separation result for disjoint closed convex sets in a finite dimensional vector space, and for this special case one can actually prove a stronger result. Since this is not well-known, an elementary proof of this stronger result is also presented in the Appendix B.

The proof of the next result follows immediately from Definition A.3 and A.4.

Lemma 1. *If the set valued mapping Φ is given by relation (2.2), then the following conditions are equivalent:*

(1) *The function $\phi_x : \mathbb{R}^{2m} \to \mathbb{R}$ is properly quasimonotone (see Definition A.3) on C.*
(2) *The mapping Φ is a KKM-mapping (see Definition A.4).*

In general it is difficult to verify that the function ϕ_x is properly quasimonotone, or equivalently (see Lemma 1), that Φ is a KKM-mapping. Therefore, a sufficient condition involving a well-known function class is given in the next lemma.

Lemma 2. *If the function $\phi_x : \mathbb{R}^{2m} \to \mathbb{R}$ satisfies $\phi_x(y, y) \geq 0$ for every $y \in C$ and $v \to \phi_x(y, v)$ is quasiconvex (see Definition A.2) on C for every $y \in C$, then the function ϕ_x is properly quasimonotone on C.*

Proof. Let $\{v_1, ..., v_k\} \subseteq C$ be given. Since the function $v \to \phi_x(y, v)$ is quasiconvex on C for every $y \in C$ it follows for every $y \in C$ that

$$\max_{1 \le i \le k} \phi_x(y, v_i) = \max_{v \in co(\{v_1,...,v_k\})} \phi_x(y, v)$$

and this implies, using $\phi_x(y, y) \ge 0$ for every $y \in C$, that

$$\max_{1 \le i \le k} \phi_x(y, v_i) = \max_{v \in co(\{v_1,...,v_k\})} \phi_x(y, v) \ge 0$$

for every y belonging to $co(\{v_1, ..., v_k\})$. Therefore we obtain that

$$\inf_{y \in co(\{v_1,...,v_k\})} \max_{1 \le i \le k} \phi_x(y, v_i) \ge 0$$

and the result is verified. □

As an immediate consequence of Lemma 1 and Theorem B.3 (or B.4) of the Appendix, we now have the following result.

Theorem 1. *Let $y \to \phi_x(y, v)$ be upper semicontinuous (see Definition A.5) for every $v \in C$, then the following statements hold:*

(1) *If the function ϕ_x is properly quasimonotone on C, then for every finite set $\{v_1, ..., v_k\} \subseteq C$ we have*

$$co(\{v_1, \cdots, v_k\}) \cap \cap_{i=1}^k \Phi(v_i) \neq \emptyset.$$

(2) *If additionally the function $y \to \phi_x(y, v)$ is quasiconcave on C for every $v \in C$, then the function ϕ_x is properly quasimonotone if and only if for every finite set $\{v_1, ..., v_k\} \subseteq C$ we have*

$$co(\{v_1, \cdots, v_k\}) \cap \cap_{i=1}^k \Phi(v_i) \neq \emptyset.$$

Proof. Since $y \to \phi_x(y, v)$ is upper semicontinuous for every $v \in C$, all its upper level sets are closed. In combination with ϕ_x being properly quasimonotone, this implies by Lemma 1 that Φ is a KKM mapping with closed values. Applying now Theorem B.3 yields the first part. To show the second part we observe that the quasiconcavity of the function $y \to \phi_x(y, v)$ on C for every $v \in C$, ensures that the set valued mapping Φ has convex values. Applying now Theorem B.4 shows the second part. □

By the above result, we know that every finite intersection $\cap_{v_i \in C} \Phi(v_i)$, is nonempty. To show that the intersection $\cap_{v \in C} \Phi(v)$ is also nonempty (or equivalently, $V(\phi_x, C) \neq \emptyset$), we need to impose a compactness-type assumption.

Theorem 2. *Suppose there exist some compact sets $B \subseteq C$ and $S \subseteq C$ satisfying*

$$\inf_{v \in B} \phi_x(y, v) < 0 \tag{2.3}$$

for every $y \in C \backslash S$. If the function $y \to \phi_x(y, v)$ is upper semicontinuous for every $v \in C$ and ϕ_x is properly quasimonotone on C, then the set $V(\phi_x, C)$ is nonempty.

Proof. Since there exist compact sets $B \subseteq C$ and $S \subseteq C$ satisfying $\inf_{v\in B} \phi_x(y,v) < 0$ for every $y \in C\backslash S$ we obtain that the set valued mapping Φ given by relation (2.2) satisfies

$$\cap_{v\in B}\Phi(v) = \{y \in C : \inf_{v\in B} \phi_x(y,v) \geq 0\} \subseteq S. \tag{2.4}$$

Moreover, using that $y \to \phi_x(y,v)$ is upper semicontinuous for every $v \in C$, we obtain that Φ has closed values and so by relation (2.4) the set $\cap_{v\in B}\Phi(v)$ is a closed subset of a compact set and hence compact. This implies that the mapping $\overline{\Phi} : C\backslash B \rightrightarrows C$ given by

$$\overline{\Phi}(v) = \Phi(v) \cap (\cap_{v\in B}\Phi(v))$$

has compact values. Since $\cap_{v\in C}\Phi(v) = \cap_{v\in C\backslash B}\overline{\Phi}(v)$, it is now sufficient, in view of the finite intersection property of compact sets (see [16]) applied to the collection $\{\overline{\Phi}(v) : v \in C\backslash B\}$, to verify that the intersection $\cap_{i=1}^{k}\overline{\Phi}(v_i)$ is nonempty for every finite collection $\{v_1,\cdots,v_k\} \subseteq C\backslash B$. To show this, let $\{v_1,\cdots,v_k\} \subseteq C\backslash B$ be given and consider an arbitrary finite set $\{v_{k+1},\cdots,v_{k+l}\} \subseteq B$. By Theorem 1, it follows that

$$co(\{v_1,...,v_{k+l}\}) \cap (\cap_{i=1}^{k+l}\Phi(v_i)) \neq \emptyset,$$

and since $\{v_1,\cdots,v_{k+l}\} \subseteq B \cup \{v_1,\cdots,v_k\}$, this implies that

$$\cap_{i=k+1}^{k+l}\Theta(v_i) \neq \emptyset\ , \tag{2.5}$$

where

$$\Theta(v) := \Phi(v) \cap (\cap_{i=1}^{k}\Phi(v_i) \cap co(B \cup \{v_1,\cdots,v_k\})).$$

Since the set B is compact, the set $co(B \cup \{v_1,\cdots,v_k\})$ is also compact, and hence for every $v \in B$, the nonempty set $\Theta(v)$ is compact. Using now again the finite intersection property for compact sets applied to the collection $\{\Theta(v) : v \in B\}$, we obtain by relation (2.5) that

$$(\cap_{i=1}^{k}\overline{\Phi}(v_i)) \cap co(B \cup \{v_1,\cdots,v_k\}) = \cap_{v\in B}\Theta(v) \neq \emptyset,$$

and we have verified the desired result. □

Remark 1. *If the set C is compact, then clearly the compactness-type assumption listed in relation (2.3) is trivially satisfied by taking $S = B = C$, and so this condition is only nontrivial for a noncompact, convex and closed set C. Moreover, it is straightforward to see that the typical compactness-type condition used in the literature (see* [8] *and references therein) does imply relation (2.3). Actually, this compactness-type condition is a generalization of a similar condition for ϕ given by (1.5) (see* [12]*).*

Before we conclude this section, we can illustrate our feasibility results on the Stampacchia variational inequalities. It is clear that the function $v \to \phi_x(y,v)$ in (1.5) is linear and the condition $\phi_x(y,y) \geq 0$ holds. Thus, by Lemma 2, ϕ_x is a properly quasimonotone function. We make the common assumptions as in the literature (see [8, 7]) and suppose that for an arbitrary x, the function $y \to F(x,y)$ is continuous and the set valued mapping K has compact convex values (or assume that the compactness-type condition (2.3) holds, see Remark 1). Then, as a direct consequence of Theorem 2, we state that there exists a feasible solution for the Stampacchia variational inequality problem. As a last note, it is well-known in the variational inequality literature that compactness-type assumptions can be further relaxed by imposing additional assumptions on the function F (see [8]).

3. Structure of the Feasible Set

Recall from (1.3) that the feasible set of an ECOP is given by

$$\mathcal{F} = Z \cap E \cap \text{graph}(K).$$

In this section we analyze the topological structure of $\mathcal{F}$ in order to state some conditions under which the intersection $E \cap \text{graph}(K)$ is closed and convex. We start with stating the conditions for closedness.

Lemma 3. *If the set valued mapping K is closed (see Definition A.6) and lower semicontinuous (see Definition A.8), and the function ϕ is upper semicontinuous, then the set $E \cap \text{graph}(K)$ is closed.*

Proof. Since the set graph(K) is closed by hypothesis, it is sufficient to show that the set E is closed. Let (x_n, y_n) belong to E and suppose (x_n, y_n) converges to (x, y). Choose any element $v \in K(x)$. Since K is lower semicontinuous it follows that one can find some sequence $v_n \in K(x_n)$ converging to v. Hence, $\phi(x_n, y_n, v_n) \geq 0$ and by the upper semicontinuity of ϕ we obtain that $\phi(x, y, v) \geq 0$. Since v is an arbitrary element of $K(x)$ this implies that $(x, y) \in E$ and the result is proved. □

In the next counterexample we illustrate that the condition for K being lower semicontinuous is crucial in the above result.

Example 1. *Consider the ECOP with $\phi(x, y, v) = (v - y)$, $K(x) = \{1\} \cup \{v : -x \leq v \leq 0\}$ where $x, y, v \in \mathbb{R}$. Then the equilibrium constraints $v - y \geq 0, \forall v \in K(x)$ lead to the condition*

$$\begin{array}{rcll} -x & \geq & y & \text{for } x \geq 0 \\ 1 & \geq & y & \text{for } x < 0 . \end{array}$$

So the points in $\{(x, y) : x = 0, 0 < y \leq 1\}$ are boundary points of E but do not belong to E and also the set $E \cap \text{graph}(K)$ is not closed:

$$E \cap \text{graph}(K) = \{(x, -x) : x \geq 0\} \cup \{(x, 1) : x < 0\}.$$

Let now K be defined explicitly by

$$K(x) = \{v \in \mathbb{R}^m : G(x, v) \leq 0\}, \tag{3.1}$$

where $G : \mathbb{R}^{n+m} \to \mathbb{R}^q$ is a continuous function and $v \to G(x, v)$ is convex for every $x \in \mathbb{R}^n$. Clearly, the graph of K becomes

$$\text{graph}(K) = \{(x, v) : v \in K(x)\} = \{(x, v) : G(x, v) \leq 0\}. \tag{3.2}$$

In this case, the set valued mapping K has closed convex values. In the next result we specify sufficient conditions for K to be lower semicontinuous.

Lemma 4. *Let the function $G : \mathbb{R}^{n+m} \to \mathbb{R}^q$ be continuous, and assume that $v \to G(x, v)$ is convex. If the set $K_0(x) := \{v \in \mathbb{R}^m : G(x, v) < 0\}$ is nonempty for every $x \in \mathbb{R}^n$ (*Slater condition*), then the set valued mapping K is lower semicontinuous.*

Proof. We will first show that the set valued mapping K_0 is lower semicontinuous. Fix $x \in \mathbb{R}^n$ and consider an arbitrary sequence x_n converging to x. For any $v \in K_0(x)$ it follows by definition that $G(x, v) < 0$ and by the continuity of G this implies that there exists some $n_0 \in \mathbb{N}$ such that $G(x_n, v) < 0$ for every $n \geq n_0$. Hence it holds that $v \in K_0(x_n)$ for every $n \geq n_0$ and so by taking $v_n = v$ for $n \geq n_0$ we have verified that K_0 is lower semicontinuous. Since the function $v \to G(x, v)$ is convex for every $x \in \mathbb{R}^n$ and $K_0(x)$ is nonempty we obtain for every $v_0 \in K_0(x)$ and $v \in K(x)$ that the convex combination $v_\lambda := \lambda v_0 + (1 - \lambda)v$ belongs to $K_0(x)$ for every $0 < \lambda < 1$. This implies that $cl(K_0(x)) = K(x)$. Using now that lower semicontinuity is preserved under taking closures we obtain that the set valued mapping K is lower semicontinuous. □

Next we study the convexity of the feasible set $\mathcal{F}$. We assume that graph(K) is convex.

Lemma 5. *If the set valued mapping K is concave and convex, and the function ϕ is quasiconcave, then the set $E \cap$ graph(K) is convex.*

Proof. The set graph(K) is convex from the hypothesis. It is now sufficient to show that the set E is convex. Let $(x_1, y_1), (x_2, y_2) \in E$ and for $\lambda \in (0, 1)$ define $x_\lambda := \lambda x_1 + (1 - \lambda)x_2$ and $y_\lambda := \lambda y_1 + (1 - \lambda)y_2$. Since the set-valued mapping K is concave, it follows for every $v \in K(x_\lambda)$ that there exists some $v_1 \in K(x_1)$ and $v_2 \in K(x_2)$, such that

$$v = \lambda v_1 + (1 - \lambda)v_2.$$

As a direct consequence of ϕ being quasiconcave, we have

$$\phi(x_\lambda, y_\lambda, v) \geq \min\{\phi(x_1, y_1, v_1), \phi(x_2, y_2, v_2)\} \geq 0 .$$

Since v is an arbitrary element of the set $K(x_\lambda)$, we conclude that (x_λ, y_λ) belongs to E. □

Notice that the conditions of Lemma 5 are rather strong. However, these assumptions are satisfied for certain applications. In the following examples K is both concave and convex.

- The mapping K is constant, *i.e.*, $K(x) = C$, $\forall x$. Then it is immediately clear that the set valued mapping is concave and convex.
- Let K be defined by

$$K(x) := \{v \in \mathbb{R}^m : G(v - Ax) \leq 0\}.$$

where $G : \mathbb{R}^m \to \mathbb{R}^q$ is convex and A an $m \times n$ matrix. Then by setting $w := v - Ax$ or $v = w + Ax$ and $C_0 := \{w \in \mathbb{R}^m \mid G(w) \leq 0\}$ we obtain

$$K(x) = \{w + Ax \mid G(w) \leq 0\} = C_0 + Ax .$$

From this representation it is obvious that K is both concave and convex.

- In Section 6 we analyze the (linear) case

$$K(x) = \{v \in \mathbb{R}^m \mid B^1 x + B^2 v \leq \beta\} \ .$$

It is not difficult to show that in this case K is both concave and convex if rank $[B^1 \ B^2] =$ rank $B^2 \leq m$ (i.e., if K is defined (essentially) by no more conditions than the dimension m).

In the linear case (Section 6) we consider functions of the form $\phi(x, y, v) = (v - y)^T \gamma$ (full linear case) and $\phi(x, y, v) = (v - y)^T(C^1 x + C^2 y + C^3 v + \gamma)$. In the first case ϕ is (trivially) quasiconcave but in the other case, except for $[C^1 \ C^2 \ C^3] = 0$, it is not.

4. Formulation of an ECOP as a Nonlinear Program

In this section we are interested in reformulations of ECOP, which are suitable for the numerical solution of the problems. We transform an ECOP to a problem with bilevel structure and obtain a formulation of the program as a nonlinear problem with complementarity constraints.

To deal with the equilibrium constraints (1.2) of ECOP, consider the optimization problem

$$(Q(x,y)) \qquad \begin{array}{ll} \min\limits_{v} & \phi(x, y, v) \\ \text{s.t.} & v \in K(x), \end{array}$$

depending on the parameter (x, y). Obviously (assuming that $Q(x, y)$ is solvable), for a solution $v = v(x, y)$ of $Q(x, y)$, we can write

(4.1)

$$E \cap \text{graph}(K) = \{(x, y) : y \in K(x) \text{ and the solution } v \text{ of } Q(x, y) \text{ satisfies } \phi(x, y, v) \geq 0\} \ .$$

Recall that the feasible set of an ECOP is given by $\mathcal{F} = Z \cap E \cap \text{graph}(K)$. So an ECOP can be written in the form

$$(P_2) \qquad \begin{array}{ll} \min\limits_{x,y,v} & f(x, y) \\ \text{s.t.} & (x, y) \in Z \\ & y \in K(x) \\ & \phi(x, y, v) \geq 0 \\ & v \text{ is a solution of } Q(x, y). \end{array}$$

Remark 2. *In view of the constraints*

$$\phi(x, y, v) \geq 0 \ \ \forall v \in K(x)$$

(if the sets $K(x)$ are infinite) formally an ECOP can be seen as a so-called generalized semi-infinite problem *(GSIP) (see* e.g. [19], [18]*). In the form P_2 it is a typical* bilevel problem *(see* e.g [4]*).*

Under the extra assumption

$$\phi(x, y, y) = 0 \ \text{ for all } y \ , \tag{4.2}$$

the parameter v in P_2 can be eliminated as follows. Condition (4.2) implies for any $y \in K(x)$:

$$\min_{v \in K(x)} \phi(x,y,v) \le \phi(x,y,y) = 0,$$

i.e., if a minimizer v of $Q(x,y)$ satisfies $\phi(x,y,v) \ge 0$ (thus $= 0$), then y must also solve $Q(x,y)$. So $E \cap \text{graph}(K) = \{(x,y) : y \in K(x),\ y \text{ is a solution of } Q(x,y)\ \}$ and P_2 simplifies:

$$(\widetilde{P}_2) \qquad \begin{array}{ll} \min\limits_{x,y} & f(x,y) \\ \text{s.t.} & (x,y) \in Z \\ & y \in K(x) \\ & y \text{ is a solution of } Q(x,y). \end{array}$$

We now assume that the sets Z and $K(x)$ are given explicitly in the form

$$Z = \{(x,y) \in \mathbb{R}^{n+m} : g(x,y) \le 0\}, \quad K(x) = \{v \in \mathbb{R}^m : G(x,v) \le 0\}$$

with C^1-functions $g : \mathbb{R}^{n+m} \to \mathbb{R}^p$ and $G : \mathbb{R}^{n+m} \to \mathbb{R}^q$. Let also ϕ be from C^1.

Let $\nabla_v \phi(x,y,v)$ and $\nabla_v G(x,v)$ denote the derivatives with respect to v. If v is a solution of $Q(x,y)$ which satisfies some *constraint qualification* (CQ) then v must necessarily satisfy the *Karush-Kuhn-Tucker conditions* (KKT conditions):

$$\begin{aligned} \nabla_v \phi(x,y,v) + \lambda^T \nabla_v G(x,v) &= 0 \\ \lambda^T G(x,v) &= 0 \end{aligned}$$

with some multiplier $0 \le \lambda \in \mathbb{R}^m$. So we can consider the following relaxation of the ECOP problem P_2.

$$(P_3) \qquad \begin{array}{ll} \min\limits_{x,y,v} & f(x,y) \\ \text{s.t.} & \phi(x,y,v) \ge 0 \\ & \nabla_v \phi(x,y,v) + \lambda^T \nabla_v G(x,v) = 0 \\ & \lambda^T G(x,v) = 0 \\ & \lambda, -g(x,y), -G(x,y), -G(x,v) \ge 0 \end{array}$$

P_3 is a relaxation of P_2 in the sense that (under CQ) the feasible set of the ECOP is contained in the feasible set of P_3. In particular, any solution (x,y,v) of P_3 with the property that v is a minimizer of $Q(x,y)$, must also be a solution of the ECOP.

In case that (4.2) holds, problem P_3 reduces to (see $\widetilde{P}_2$):

$$(\widetilde{P}_3) \qquad \begin{array}{ll} \min\limits_{x,y} & f(x,y) \\ \text{s.t.} & \nabla_v \phi(x,y,y) + \lambda^T \nabla_v G(x,y) = 0 \\ & \lambda^T G(x,y) = 0 \\ & \lambda, -g(x,y), -G(x,y) \ge 0 \end{array}$$

Convexity conditions for $Q(x,y)$. Let us now consider the special case that $Q(x,y)$ represents a convex problem, *i.e.*, for any fixed x and y the function $\phi(x,y,v)$ is convex in v, and for any fixed x, the function $G(x,v)$ is convex in v. Then, it is well-known that the KKT conditions at v are sufficient for v to be a solution of $Q(x,y)$. So in this case any solution (x,y) of P_3 (or $\widetilde{P}_3$) provides a solution of an ECOP. If moreover CQ is satisfied for $Q(x,y)$ (which is automatically fulfilled if $v \to G(x,v)$ is linear), then P_3 (or $\widetilde{P}_3$) is equivalent with the original ECOP.

In the form P_3 and $\widetilde{P}_3$, an ECOP is transformed into a nonlinear program with complementarity constraints (see *e.g.*, [17]). In this form the problems can be solved numerically, for instance by an interior point method (see *e.g.*, [24]).

The linear case. In the next section we will analyze ECOP for the case that all problem functions are linear, $f(x,y) = c^1 x + c^2 y$, and

$$g_i(x,y) = a_i^1 x + a_i^2 y \leq \alpha_i, \;\; i \in I, \qquad G_j(x,y) = b_j^1 x + b_j^2 y \leq \beta_j, \;\; j \in J \; .$$

Here and in the rest of the paper we omit the transposed sign in the inner products, *i.e.*, ax denotes $a^T x$. For the function $\phi(x,y,v) = (v-y)F(x,y,v)$, we consider the case

$$\phi(x,y,v) = (v-y)(C^1 x + C^2 y + C^3 v + \gamma)$$

with matrices and vectors of obvious dimension. We assume that the ($m \times m$) matrix C^3 is positive semi-definite. Then the problem $Q(x,y)$ is convex and by the discussions above, ECOP and $\widetilde{P}_3$ are equivalent. By replacing $C^2 y + C^3 y$ by $C^2 y$ (for notational simplicity) our problem $\widetilde{P}_3$ takes the form

$$(L_{\text{ECOP}}) \qquad \begin{array}{ll} \min_{x,y} & c^1 x + c^2 y \\ \text{s.t.} & a_i^1 x + a_i^2 y \leq \alpha_i, \;\; i \in I := \{1,\dots,p\} \\ & b_j^1 x + b_j^2 y \leq \beta_j, \;\; j \in J := \{1,\dots,q\} \\ & C^1 x + C^2 y + \gamma + \sum\limits_{j \in J(x,y)} \lambda_j b_j^2 = 0 \\ & \lambda_j \geq 0, \;\; j \in J(x,y) \end{array}$$

where for $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$, we define the active index sets $J(x,y) := \{j \in J : b_j^1 x + b_j^2 y = \beta_j\}$ and also $I(x,y) := \{i \in I : a_i^1 x + a_i^2 y = \alpha_i\}$.

Remark 3. *For the special case $F(x,y,v) = \gamma$, i.e., $C^1, C^2 = 0$, the problem ECOP, or equivalently $\widetilde{P}_3$, can be written as a common* linear bilevel problem

$$\begin{array}{ll} \min_{x,y} & c^1 x + c^2 y \\ \textit{s.t.} & a_i^1 x + a_i^2 y \leq \alpha_i, \;\; i \in I \\ & y \textit{ is a solution of } Q(x,y): \\ & \quad \min \;\; \gamma v - \gamma y \\ & \quad \textit{s.t.} \;\;\; b_j^1 x + b_j^2 v \leq \beta_j, \;\; j \in J, \end{array}$$

and L_{ECOP} becomes

$$(L_{\text{BL}}) \qquad \begin{array}{ll} \min_{x,y} & c^1 x + c^2 y \\ \textit{s.t.} & a_i^1 x + a_i^2 y \leq \alpha_i, \;\; i \in I \\ & b_j^1 x + b_j^2 y \leq \beta_j, \;\; j \in J \\ & \gamma + \sum\limits_{j \in J(x,y)} \lambda_j b_j^2 = 0, \\ & \lambda_j \geq 0, \;\; j \in J(x,y). \end{array}$$

So for this special case the third constraints become 'independent' from the other constraints which means that L_{ECOP} has a more complicated structure than the bilevel problem L_{BL}.

In [20] a genericity analysis was done for linear bilevel (i.e. for the case L_{BL}). Note that also the (full) linear case $\phi(x,y,v) = ax + by + cv$ leads (via P_3) to a problem of bilevel structure. In the next section we are going to analyze the structure of L_{ECOP} from a generic point of view (structure in the general case).

5. The Generic Structure of Linear ECOP

In the present section we reconsider the (linear) ECOP of the form L_{ECOP}. We are going to analyze the structure of L_{ECOP} from a generic point of view (structure in the general case). In [20] a genericity analysis was done for the linear bilevel problems L_{BL}, which corresponds to the case $[C_1\ \ C_2] = 0$ (see Remark 1). Since both problems L_{BL} and L_{ECOP} have a similar structure, the genericity analysis for L_{ECOP} can be performed with similar techniques. We therefore present the results here in a concise form but emphasize that the more general problem L_{ECOP} leads to a more complicated structure of the feasible set than problem L_{BL}.

First we introduce some abbreviations

$$z = (x, y),\ \ c = (c^1, c^2),\ \ a_i = (a_i^1, a_i^2),\ \ b_j = (b_j^1, b_j^2) \in \mathbb{R}^{n+m} \text{ and } C = (C^1, C^2).$$

We define the matrices A, B, B^2 with rows $a_i, i \in I$, $b_j, j \in J$, $b_j^2, j \in J$, respectively, and for the vectors $\alpha = (\alpha_1, \dots, \alpha_p)$, $\beta = (\beta_1, \dots, \beta_q)$, we also introduce the *constraint sets*

$$Q_A = \{z : Az \le \alpha\},\ \ Q_B = \{z : Bz \le \beta\},\ \ Q = Q_A \cap Q_B.$$

This leads to the following compact form

$$(L_{\text{ECOP}}) \qquad \begin{array}{ll} \min & cz \\ \text{s.t.} & Az \le \alpha \\ & Bz \le \beta \\ & Cz + \gamma + \sum\limits_{j \in J(x,y)} \lambda_j b_j^2 = 0 \\ & \lambda_j \ge 0,\ \ j \in J(x,y). \end{array}$$

Note that if we assume that Q is compact (bounded) and that the feasible set of L_{ECOP} is non-empty, it is clear that a solution always exists.

For linear bilevel problems, the feasible set simply consists of a union of faces (of dimension n) of the polyhedron Q. Moreover, for the special case $I = \emptyset$, the feasible set (in general non-convex) is (path-)connected. Both facts are no more true for L_{ECOP}.

Genericity. For fixed problem parameters (n, m, p, q), any L_{ECOP} can be seen as an element from the *problem set*

$$\mathcal{P} = \{P = (c, A, B, \alpha, \beta, C, \gamma)\} \equiv \mathbb{R}^K \text{ with } K = n + (n + m + 1)(m + p + q).$$

Throughout the paper, by a generic subset $\mathcal{P}_0$ of $\mathcal{P} \equiv \mathbb{R}^K$ we mean a set, which is open in $\mathbb{R}^K$ and has a complement set of measure zero (notation $\mu(\mathbb{R}^K \setminus \mathcal{P}_0) = 0$). Note that this implies that the set $\mathcal{P}_0$ is dense in $\mathbb{R}^K$. For details on genericity we refer to [6] and [11].

Our genericity analysis will be based on the following 'non-trivial' result (see [6]).

Lemma 6. *Let $p : \mathbb{R}^K \to \mathbb{R}$ be a polynomial function, $p \not\equiv 0$. Then, the solution set $p^{-1}(0) = \{w \in \mathbb{R}^K \mid p(w) = 0\}$ is a closed set of measure zero. Equivalently the complement $G = \mathbb{R}^K \setminus p^{-1}(0)$ is a generic set in $\mathbb{R}^K$.*

Remark 4. *The result of Lemma 6 will be used repeatedly as follows. By noticing that $\det A = \sum_{\pi\in\Pi_l}$ signπ $a_{1\ \pi(1)} \cdots a_{l\ \pi(l)}$ defines a polynomial mapping $p : \mathbb{R}^{l\cdot l} \to \mathbb{R}$ we directly are led to the following result: Let V_l denote the set of real $(l \times l)$-matrices, $V_l = \{A = (a_{ij})_{i,j=1,\ldots,l} \mid a_{ij} \in \mathbb{R}\} \equiv \mathbb{R}^{l\times l}$. Then, the set $V_l^0 = \{A \in V_l \mid \det A = 0\}$ is a closed set of measure zero in $\mathbb{R}^{l\times l}$. Equivalently the set $V_l^r = V_l \setminus V_l^0$ of regular matrices is generic in $\mathbb{R}^{l\cdot l}$.*

In the sequel, $z_0 = (x_0, y_0)$ will be a point such that with appropriate multipliers $\lambda_j,\ j \in J(z_0)$, the constraints of L_{ECOP} are fulfilled. We then call z_0 or (z_0, λ) a feasible point for L_{ECOP}. Often the abbreviation $I_0 = I(z_0),\ J_0 = J(z_0)$ will be used.

We say that at a feasible point (z_0, λ) the *strict complementary slackness condition holds* if for all $j \in J$:

(SC) $$\lambda_j > 0 \Leftrightarrow (\beta_j - b_j z_0) = 0.$$

Among others it will be analyzed whether generically the condition SC holds at a solution of L_{ECOP}. The answer will be negative.

Remark 5. *For the special case that Q_A is contained in the interior of Q_B (implying $Q = Q_A$) our problem takes the form of a common LP:*

(L_{ECOP}) $$\begin{aligned} \min \quad & cz \\ & Az \le \alpha \\ & Cz = -\gamma. \end{aligned}$$

Here, the generic structure is simply given by the well-known generic structure of such an LP.

We now are going to analyze the structure of the feasible set of L_{ECOP} near a feasible point (z_0, λ_0) and define

$$J_0^a = \{j \in J_0 : [\lambda_0]_j = 0\} \text{ and } J_0^n = J_0 \setminus J_0^a.$$

The following observation is crucial for the analysis below. Since the vector $-(Cz_0 + \gamma) \in \mathbb{R}^m$ is an element of cone $\{b_j^2,\ j \in J_0^n\}$ by Caratheodory's theorem we can assume

(5.1) $$|J_0^n| \le m.$$

Consider now a feasible direction d_0 at (z_0, λ_0) given by a solution (d_0, δ_0) of the system:

(5.2) $$\begin{aligned} a_i d &\le 0,\ i \in I_0 \\ b_j d &\le 0,\ j \in J_0^a \\ b_j d &= 0,\ j \in J_0^n \\ Cd + \sum_{j\in J_0} \delta_j b_j^2 &= 0, \\ \delta_j (b_j d) &= 0,\ j \in J_0^a \\ \delta_j &\ge 0. \end{aligned}$$

The following necessary condition for local minimizers is obvious.

Lemma 7. *Let (z_0, λ_0) be feasible for L_{ECOP}. Then, if z_0 is a local minimizer, there is no solution (d, δ) of (5.2) such that $cd < 0$,* i.e., *there is no feasible descent direction.*

Note that for any solution (d, δ) of (5.2) the points $(z(t), \lambda(t)) = (z_0 + td, \lambda_0 + t\delta)$ are feasible for L_{ECOP} if $t \geq 0$ is not too large. As a first genericity result we obtain the following lemma.

Lemma 8. *Generically for any local solution z_0 of L_{ECOP} the condition $|I(z_0)| + |J(z_0)| \geq n$ must hold.*

Proof. Suppose that $|I_0| + |J_0| < n$ ($I_0 = I(z_0), J_0 = J(z_0)$). We will show that generically this implies that there is a solution (d, δ) of (5.2) satisfying $cd < 0$ and the result follows by Lemma 6. To do so consider the system

$$\begin{aligned} cd &= -1 \\ a_i d &= 0, \qquad i \in I_0 \\ b_j d &= 0, \qquad j \in J_0 \\ Cd + \sum_{j \in J_0} \delta_j b_j^2 &= 0 \\ \delta_j &= 1, \qquad j \in J_0 \end{aligned}$$

with $s := 1 + |I_0| + |J_0| + m + |J_0|$ equations in $n + m + |J_0| \geq s$ unknowns. Generically the system matrix has full rank (see Remark 4) and thus admits a solution. □

Noticing that y_0 is a boundary point of $K(x_0)$ if and only if $J(x_0, y_0) \neq \emptyset$, we obtain the following result as a corollary.

Corollary 1. *Generically for any local minimizer $z_0 = (x_0, y_0)$ of L_{ECOP} which satisfies $|I(z_0)| < n$, y_0 must be a boundary point of $K(x_0)$.*

The next theorem states that in the generic case the feasible set of an L_{ECOP} is n-dimensional (in the z-space).

Theorem 3. *Generically the (projection onto the z-space of the) feasible set of L_{ECOP} consists of a (finite) union of polyhedra of dimension n.*

Proof. Let be given (z_0, λ_0), feasible for L_{ECOP} with corresponding index sets I_0, J_0, J_0^a, J_0^n, $|J_0^n| \leq m$ (see (5.1)). We will show that generically near z_0 the feasible set (in the z-space) has exactly dimension n.

<u>dimension at most n:</u> Any feasible point (z, λ) must be a solution of an equation

$$\begin{aligned} b_j z = \beta_j, \quad j \in J_0^n \\ Cz + \sum_{j \in J_0^n} \lambda_j b_j^2 = -\gamma \end{aligned}$$

for some subset $J_0^n \subset J$ with $|J_0^n| \leq m$. Generically this system has full rank $|J_0^n| + m$ and thus its solution set is of dimension $n + m + |J_0^n| - m - |J_0^n| = n$ in the (z, λ)-space. Consequently its dimension in the z-space (projection) cannot exceed n.

dimension at least n: Note first that (z_0, λ_0) is a solution of the equations

$$\begin{aligned} a_i z &= \alpha_i, \quad i \in I_0 \\ b_j z &= \beta_j, \quad j \in J_0 \\ Cz + \sum_{j \in J_0^n} \lambda_j b_j^2 &= -\gamma. \end{aligned} \tag{5.3}$$

Generically this system has full rank

$$k = \min\{|I_0| + |J_0| + m, n + m + |J_0^n|\}$$

with $|J_0^n| \leq m$. Moreover, the system of $n + m + |J_0^n|$ unknowns must satisfy the relation

$$|I_0| + |J_0| + m \leq n + m + |J_0^n| \text{ or equivalently } |I_0| + |J_0^a| \leq n. \tag{5.4}$$

To see this assume that $|I_0|+|J_0|+m \geq n+m+|J_0^n|+1$, then the vector $(\alpha, \beta, -\gamma) \in \mathbb{R}^{|I_0|+|J_0|+m}$ (right-hand side of (5.3)) is contained in the $(n + m + |J_0^n|)$-dimensional space spanned by the columns of the system matrix in (5.3), (a closed set of measure zero in $\mathbb{R}^{|I_0|+|J_0|+m}$). This is generically excluded.

Consider now the system

$$\begin{aligned} a_i d &= -1, \quad i \in I_0 \\ b_j d &= 0, \quad j \in J_0^n \\ b_j d &= -1, \quad j \in J_0^a \\ Cd + \sum_{j \in J_0^n} \delta_j b_j^2 &= 0. \end{aligned}$$

Since generically $|I_0| + |J_0^a| \leq n$ must hold (see (5.4)) this is a system of $|I_0| + |J_0| + m \leq n + m + |J_0^n|$ equations in $n + m + |J_0^n|$ unknowns. So generically there is a solution (d, δ) of this system (possibly zero in the case $I_0 = J_0^a = \emptyset$). By construction, for any $t_1 > 0$ small enough, the point

$$(z_1, \lambda_1) = (z_0, \lambda_0) + t_1(d, \delta)$$

is feasible for L_{ECOP} with $I(z_1) = \emptyset, J(z_1) = J_0^n$ ($[\lambda_1]_j > 0,\ j \in J(z_1)$). Consequently, near (z_1, λ_1) all points $(z, \lambda) = (z_1, \lambda_1) + t(d, \delta)$, $t > 0$ (small) are feasible if (d, δ) solves the equations

$$\begin{aligned} b_j d &= 0, \quad j \in J_0^n \\ Cd + \sum_{j \in J_0^n} \delta_j b_j^2 &= 0. \end{aligned} \tag{5.5}$$

This system of $|J_0^n| + m$ equations generically has a solution set of dimension

$$n + m + |J_0^n| - |J_0^n| - m = n$$

in the (z, λ)-space. But generically also the projection of this solution set to the z-space is of dimension n. To see this, consider the system (5.5). Since $|J_0^n| \leq m$ we can decompose the system as

$$\begin{pmatrix} B & 0 \\ C_1 & B_1^2 \\ C_2 & B_2^2 \end{pmatrix} \begin{pmatrix} d \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

with a $|J_0^n| \times |J_0^n|$-matrix B_2^2, which is generically regular. From the last $|J_0^n|$ equations we can eliminate δ,

$$\delta = -(B_2^2)^{-1} C_2 d$$

resulting in the system

$$\begin{aligned} Bd &= 0 \\ (C_1 - B_1^2(B_2^2)^{-1}C_2)d &= 0 \end{aligned}$$

with m equations for the $n+m$ unknowns. With the help of Lemma 6 it is not difficult to show that also this system generically has full rank m, *i.e.*, generically the solution space has dimension $n+m-m=n$. □

Remark 6. *More precisely, according to the proof of Theorem 3, generically, the feasible set (projected onto the z-space) of L_{ECOP} has the following structure. The polyhedron Q is generically either empty or has full dimension $n+m$. So L_{ECOP} consists of the sub-polyhedron $\{z \in Q : Cz+\gamma=0\}$ (generically empty or n-dimensional) together with a (finite) union of n-dimensional sub-polyhedra on faces defined by the equalities $b_jz=\beta_j$. Note that by convexity, each of these faces can only contain one of these feasible polyhedra.*

Finally, by a simple example we show that, in case $I=\emptyset$, in contrast to L_{BL} (see Remark 4 and [20]), the feasible set of L_{ECOP} need not be connected.

Example 2. *Consider the L_{ECOP} with $n=m=1$ and the feasible set defined by ($(z=(x,y))$*

$$\begin{aligned} b_jz &\le \beta_j, \;\; j \in J := \{1,2,3,4\} \\ Cz+\gamma &= -\textstyle\sum_{j\in J(z)} \lambda_j b_j^2 \,. \end{aligned}$$

The feasible set is given by the points in $Q := \{z \mid b_jz \le \beta_j,\; j=1,\dots,4\}$ which satisfy one of the relations $Cz=-\gamma$ or

$$\begin{aligned} b_jz &= \beta_j \\ Cz+\gamma &= -\lambda b_j^2, \quad \lambda \ge 0\,, \end{aligned} \tag{5.6}$$

for the indices $j \in J$. The structure of the feasible set depends on the choice of the data C, b_1 etc. Let us now choose $C=(0,-1),\; \gamma=0,\; \beta_1=\beta_2=\beta_3=\beta_4=1$ and

$$b_1=(0,1),\; b_2=(-1,1/2),\; b_3=(1,1/2),\; b_4=(0,-1)\,.$$

Then the feasible set consists of the set $F_0=\{z=(x,y)\in Q \mid Cz=-\gamma\}=\{(x,0)\mid -1\le x\le 1\}$ and the parts on the faces of Q given by (5.6) for $j=1,\dots,4$:

$$\begin{aligned} F_1 &= \{z=(x,y)\in Q \mid b_1z=1,\; Cz=-\lambda b_1^2, \lambda\ge 0\} = \{(x,1)\mid -1/2\le x\le 1/2\} \\ F_2 &= \{(\tfrac{1}{2}y-1,y)\mid 0\le y\le 1\} \\ F_3 &= \{(1-\tfrac{1}{2}y,y)\mid 0\le y\le 1\} \\ F_4 &= \{(x,-1)\mid -1.5\le x\le 1.5\} \end{aligned}$$

So obviously, the feasible set $F=\cup_{j=0}^4 F_j$ is not connected. Note that this situation is stable with respect to (small) perturbations of the parameter values.

We end up with an observation which is important from a theoretical and practical point of view. For any given subset $J_0 \subset J$ we consider the LP:

$$(P(J_0)) \qquad \begin{array}{ll} \min & cz \\ \text{s.t.} & Az \leq \alpha \\ & Bz \leq \beta \\ & b_j z = \beta_j, \;\; j \in J_0 \\ & Cz + \gamma + \sum\limits_{j \in J_0} \lambda_j b_j^2 = 0, \\ & \lambda_j \geq 0, \;\; j \in J_0. \end{array}$$

So obviously, to solve L_{ECOP} amounts to solving the problem:

- Find the index set J_0 ($J_0 \subset J$) such that the objective value of $P(J_0)$ is minimal.

In a forthcoming paper we describe a *descent method* which by updating J_0 in each step finds a local minimizer of L_{ECOP}. With regard to the problem $P(J_0)$ we can directly deduce the following

- Generically, every point z_0 in Q , *i.e.*, every feasible point of L_{ECOP}, satisfies $|I(z_0)| + |J(z_0)| \leq n + m$.
- Generically each problem $P(z_0)$ attains a (unique) solution at a (non-degenerate) vertex (z_0, λ_0) of the corresponding polyhedron. In particular $n + m + |J_0|$ constraints must be active. This implies that precisely for $n - |I(z_0)|$ indices $j \in J$, either $\lambda_j = 0$ for $j \in J_0$ must be active, or $b_j z_0 = \beta_j$, for $j \in J \setminus J_0$. So in the extreme case $I = \emptyset$ the (SC) condition is violated for n indices.

Conclusion.
This paper studies a form of an *equilibrium constrained optimization problem* (ECOP) which contains *bilevel programs* (BL) and *generalized semi-infinite problems* (GSIP) as special instances. The relation and differences between these three types of problems is analysed. Based on the KKM lemma, under certain convexity assumptions, the existence of feasible points can be proven. For a special linear ECOP a full genericity analysis is given which constitutes the basis for efficient algorithms to compute (local) minimizer of ECOP.

APPENDIX A

We refer to [3] for generalized convexity related definitions and for definitions from set valued-analysis we refer to [2].

Definition A.1. *A function $\psi : \mathbb{R}^m \to \mathbb{R}^m$ is called pseudomonotone if for every $x, y \in \mathbb{R}^m$*

$$\langle \psi(x), x - y \rangle \geq 0 \textit{ implies that } \langle \psi(y), x - y \rangle \geq 0.$$

Definition A.2. *A function $\psi : \mathbb{R}^n \to \mathbb{R}$ is called quasiconvex if all its sublevel sets are convex. A function ψ is quasiconcave if $-\psi$ is quasiconvex.*

Definition A.3. *A function $\psi : \mathbb{R}^{2m} \to \mathbb{R}$ is called properly quasimonotone on the convex set $X \subseteq \mathbb{R}^m$ if*

$$\inf_{y\in co(\{x_1,\dots,x_k\})} \max_{1\le i\le k} \psi(y, x_i) \ge 0$$

for every finite set $\{x_1, ..., x_k\} \subseteq X$.

Definition A.4. *A set valued mapping $\Psi : X \rightrightarrows X$ is called a KKM-mapping if*

$$co(\{x_1, ..., x_k\}) \subseteq \cup_{i=1}^k \Psi(x_i)$$

for every finite set $\{x_1, ..., x_k\} \subseteq X$.

Definition A.5. *A function $\psi : X \to X$ is called upper semicontinuous if all its upper level sets are closed. Similarly, it is called lower semicontinuous if all its lower level sets are closed.*

Definition A.6. *A set valued mapping $\Psi : X \rightrightarrows X$ is called closed if the set graph(Ψ) is closed.*

Definition A.7. *A set valued mapping $\Psi : X \rightrightarrows X$ is convex if and only if*

$$\lambda\Psi(x_1) + (1 - \lambda)\Psi(x_2) \subseteq \Psi(\lambda x_1 + (1 - \lambda)x_2)$$

for every $x_1, x_2 \in X$ and $0 \le \lambda \le 1$. Accordingly, we call a set-valued mapping Ψ concave if

$$\Psi(\lambda x_1 + (1 - \lambda)x_2) \subseteq \lambda\Psi(x_1) + (1 - \lambda)\Psi(x_2)$$

for every $x_1, x_2 \in X$ and $0 \le \lambda \le 1$.

Definition A.8. *A set valued mapping $\Psi : X \rightrightarrows X$ is called lower semicontinuous at $x \in X$ if for every $v \in \Psi(x)$ and for every sequence x_n converging to x, there exists a sequence $v_n \in \Psi(x_n)$, such that v_n converges to v. Ψ is called lower semicontinuous if it is lower semicontinuous at every $x \in X$.*

APPENDIX B

To show that under certain conditions the intersection in relation (2.2) is nonempty, we apply the important KKM lemma from nonlinear analysis. Before introducing this lemma, let e_i be the i^{th} unit vector in $\mathbb{R}^n, i = 1, ..., n$ and introduce for every subset $J \subseteq N := \{1, ..., n\}$ the simplex Δ_J, given by

$$\Delta_J := co(\{e_j : j \in J\}) \subseteq \mathbb{R}^n. \tag{5.7}$$

Definition B.1. *The collection of sets $E_j \subseteq \mathbb{R}^n, 1 \le j \le n$ satisfies the KKM property if for every subset $J \subseteq \{1, ..., n\}$ it holds that $\Delta_J \subseteq \cup_{j\in J} E_j$.*

The KKM lemma is now given by the following result (*cf.* [25], [13], [5]).

Theorem B.1. *If $E_i \subseteq \mathbb{R}^n$, $i = 1, ..., n$ are closed sets satisfying the KKM property, then it follows that $\cap_{i=1}^n E_i \ne \emptyset$.*

The KKM lemma is equivalent with Sperner's lemma (see [23]) and Sperner's lemma can be proved by combinatorial arguments (*cf.* [1] or Theorem 3.4.3 of [21]). If the sets $E_i, 1 \leq i \leq n$, are additionally convex, then an elementary proof of the KKM lemma can be given (see Theorem B.2) by using the next result of Berge (*cf.* [22]). The result of Berge is based on the well-known separating hyperplane result for disjoint finite dimensional compact convex sets and its proof can be found in [22].

Lemma B.1. *If $C_i \subseteq \mathbb{R}^n, 1 \leq i \leq r$ and $r \geq 2$ are closed convex sets satisfying $\cup_{i=1}^r C_i$ is convex and for any $J \subseteq \{1, ..., r\}$ with $|J| = r - 1$ it holds that $\cap_{j \in J} C_j$ is nonempty, then it follows that $\cap_{i=1}^r C_i$ is nonempty.*

Before giving a proof of an improvement of the KKM lemma for closed convex sets based on Lemma B.1, we introduce the following definition.

Definition B.2. *The collection of sets $E_i \subseteq \mathbb{R}^n, 1 \leq i \leq n$, satisfies the simplex finite intersection property if for every subset $J \subseteq N := \{1, ..., n\}$ it holds that $\Delta_J \cap (\cap_{j \in J} E_j) \neq \emptyset$.*

For convex sets one can now give the following improvement of the KKM lemma by elementary methods. This proof is adapted from the proof of a related result in [10].

Theorem B.2. *If $E_i \subseteq \mathbb{R}^n, 1 \leq i \leq n$, is a collection of closed convex sets the following conditions are equivalent:*

(1) *The collection $E_i, 1 \leq i \leq n$, satisfies the simplex finite intersection property.*
(2) *The collection $E_i, 1 \leq i \leq n$, satisfies the KKM property.*

Proof. To prove the implication $2 \Rightarrow 1$ we verify by induction that for every $r \leq n$ and $J \subseteq \{1, ..., n\}$ satisfying $|J| \leq r$ it holds that

$$\Delta_J \cap (\cap_{j \in J} E_j) \neq \emptyset, \tag{5.8}$$

if the collection $E_i, 1 \leq i \leq n$, satisfies the KKM property. Since the KKM property holds it follows that $e_j \in E_j$ and so relation (5.8) holds for $r = 1$. Suppose now that relation (5.8) holds for $r = l - 1$ and consider a subset $J \subseteq N := \{1, ..., n\}$ consisting of l elements. Since the sets $E_j, j \in J$ are closed and convex also the nonempty sets $E_j \cap \Delta_J$, $j \in J$ are closed and convex. By the KKM property we obtain $\Delta_J \subseteq \cup_{j \in J} E_j$ and this implies

$$\cup_{j \in J}(E_j \cap \Delta_J) = \Delta_J. \tag{5.9}$$

Moreover, it follows by the induction hypothesis for every $\bar{j} \in J$ that the set $\Delta_{J/\{\bar{j}\}} \cap (\cap_{j \in J/\{\bar{j}\}} E_j)$ is nonempty and since clearly

$$\Delta_{J/\{\bar{j}\}} \cap (\cap_{j \in J/\{\bar{j}\}} E_j) \subseteq \cap_{j \in J/\{\bar{j}\}}(E_j \cap \Delta_J)$$

we obtain for every $\bar{j} \in J$ that

$$\cap_{j \in J/\{\bar{j}\}}(E_j \cap \Delta_J) \neq \emptyset. \tag{5.10}$$

Using now relations (5.9) and (5.10) we may apply Berge's lemma with C_i replaced by $E_i \cap \Delta_J$ and this shows $\Delta_J \cap (\cap_{j\in J} E_j) \neq \emptyset$ completing the induction step. To show the implication $1 \Rightarrow 2$ we need to verify for $E_i, 1 \leq i \leq n$ satisfying the simplex finite intersection property that for any subset $J \subseteq N := \{1,..,n\}$ with $|J| \leq r$ and $1 \leq r \leq n$ it follows that

$$\Delta_J \subseteq \cup_{j\in J} E_j \ . \tag{5.11}$$

If $r = 1$, then $J \subseteq N := \{1,...,n\}$ consists of one element j and so by the simplex finite intersection property we obtain that

$$e_j = \Delta_J \in E_j \ ,$$

showing that relation (5.11) holds for $r = 1$. Suppose now relation (5.11) holds for any subset J with $|J| \leq r-1$ and let $x \in \Delta_J$ with $|J| = r$. This means $x = \sum_{j\in J} \lambda_j e_j$ with $\lambda_j \geq 0$ and $\sum_{j\in J} \lambda_j = 1$. If some λ_j equals 0 we may apply the induction hypotheses and so without loss of generality we may assume that $\lambda_j > 0$ for every $j \in J$. Since the collection $E_i, 1 \leq i \leq n,$ satisfies the simplex finite intersection property it follows that there exists some nonnegative sequence $\mu_j, j \in J$ satisfying $\sum_{j\in J} \mu_j = 1$ and

$$\overline{x} := \sum\nolimits_{j\in J} \mu_j e_j \in \cap_{j\in J} E_j. \tag{5.12}$$

Introducing now the finite number

$$\nu := \max\{\mu_j \lambda_j^{-1} : j \in J\}$$

we obtain using $\mu, \lambda \in \Delta_J$ that $\nu \geq 1$. If $\nu = 1$ this implies that $\mu_j = \lambda_j$ for every $j \in J$ and so by relation (5.12) it follows that $x = \overline{x} \in \cup_{j\in J} E_j$ and we are done. Therefore $\nu > 1$ and consider now

$$\lambda_j^* := \frac{\lambda_j - \nu^{-1}\mu_j}{1-\nu^{-1}}, j \in J \ .$$

By the definition of ν we obtain $\sum_{j\in J} \lambda_j^* = 1$ and $\lambda_j^* \geq 0$. Since $\lambda_j^* = 0$ for some $j \in J$ it follows by our induction hypothesis that

$$x^* := \sum\nolimits_{j\in J} \lambda_j^* e_j \in E_{j^*}$$

for some $j^* \in J$. Moreover, by relation (5.12) we obtain $\overline{x} \in E_{j^*}$ and since $x = \nu^{-1}\overline{x} + (1-\nu^{-1})x^*$ it follows by the convexity of E_{j^*} that $x \in E_{j^*} \subseteq \cup_{j\in J} E_j$. This completes the induction step. □

We will now extend the KKM lemma to set valued mappings $\Psi : C \rightrightarrows C$ with nonempty values.

Definition B.3. *The set valued mapping $\Psi : C \rightrightarrows C$ is called a KKM mapping if $co(\{v_1,...,v_k\}) \subseteq \cup_{j=1}^k \Psi(v_j)$ for every finite subset $\{v_1,...,v_k\}$ of the set C.*

An important consequence of the KKM lemma to set valued mappings is given by the following result.

Theorem B.3. *If $\Psi : C \rightrightarrows C$ is a set valued KKM mapping with closed values, then it follows for every finite set $\{v_1,...,v_k\} \subseteq C$ that*

$$co(\{v_1,...,v_k\}) \cap (\cap_{j=1}^k \Psi(v_j)) \neq \emptyset.$$

Proof. Introduce for every $1 \leq i \leq k$ the sets $E_i := \{\lambda \in \Delta_N : \sum_{j=1}^k \lambda_j v_j \in \Psi(v_i)\}$. Since the sets $\Psi(v_i)$, $i = 1,..,k$ are closed, it follows that the sets $E_i \subseteq \mathbb{R}^n$ are also closed. Moreover, if $J \subseteq \{1,...,k\}$ and $\lambda := (\lambda_1,...,\lambda_k) \in \Delta_J \subseteq \mathbb{R}^n$ we obtain, using $co(\{v_j : j \in J\}) \subseteq \cup_{j\in J}\Psi(v_j)$, that

$$\sum\nolimits_{j=1}^k \lambda_j v_j = \sum\nolimits_{j\in J} \lambda_j v_j \in \cup_{j\in J}\Psi(v_j).$$

This shows that λ belongs to $\cup_{j\in J}E_j$ and so $\Delta_J \subseteq \cup_{j\in J}E_j$. Applying now the KKM lemma yields the desired result. □

If the set valued mapping $\Psi : C \rightrightarrows C$ has closed convex values one can show the following improvement of Theorem B.3.

Theorem B.4. *If $\Psi : C \rightrightarrows C$ is a set valued mapping with closed convex values, then it follows that Ψ is a KKM mapping if and only if for every finite set $\{v_1,...,v_k\} \subseteq C$ it holds that*

$$co(\{v_1,...,v_k\}) \cap (\cap_{j=1}^k \Psi(v_j)) \neq \emptyset.$$

Proof. If Ψ is a KKM mapping we obtain by Theorem B.3 the desired result. To prove the reverse implication we adapt in an obvious way the proof of Theorem B.2. □

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