Examination: Mathematical Programming I (158025)

June 30, 2003, 13.30-16.30

Ex.1 Prove the following statements.

(a) Let $\mathbf{A} \in \mathbb{R}^{n\times m}$ be a given matrix ($m \leq n$). Then $\mathbf{A}^T\mathbf{A}$ is positive definite if and only if $\mathbf{A}$ has full rank (m).

(b) For a symmetric $(n \times n)$-matrix $\mathbf{A}$ the following holds: $\mathbf{A}$ is positive semidefinite if and only if all eigenvalues λ_j of $\mathbf{A}$ are non-negative ($\lambda_j \geq 0, \;\; j = 1, \ldots, n$).

Ex.2 Consider the primal-dual pair of linear problems:

$$
\begin{array}{llll}
(P) & \max\limits_{\mathbf{x}\in\mathbb{R}^n} \mathbf{c}^T\mathbf{x} & \text{s.t.} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\
(D) & \min\limits_{\mathbf{y}\in\mathbb{R}^m} \mathbf{b}^T\mathbf{y} & \text{s.t.} & \mathbf{A}^T\mathbf{y} = \mathbf{c}\,, \quad \mathbf{y} \geq \mathbf{0}
\end{array}
$$

Show the following:

(a) There exists a feasible point for D (i.e. a point satisfying $\mathbf{A}^T\mathbf{y} = \mathbf{c}\,,\; \mathbf{y} \geq \mathbf{0}$) if and only if $\mathbf{c}^T\mathbf{x} \leq 0$ is implied by $\mathbf{A}\mathbf{x} \leq \mathbf{0}$.

(b) Let the feasible set $\mathcal{F}_P = \{\mathbf{x} \mid \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$ be non-empty. Show:
The value $f(\mathbf{x}) = \mathbf{c}^T\mathbf{x}$ is bounded from above on $\mathcal{F}_P$ if and only if $\mathbf{c}^T\mathbf{x} \leq 0$ is implied by $\mathbf{A}\mathbf{x} \leq \mathbf{0}$.

Ex. 3

(a) Show that $f(\mathbf{x}) = \|\mathbf{x}\|$ ($\|\mathbf{x}\|$ any norm on $\mathbb{R}^n$) defines a convex function $f : \mathbb{R}^n \to \mathbb{R}$.

(b) Let $g : \mathbb{R}^n \to I,\; I \subset \mathbb{R}$ be convex and $f : I \to \mathbb{R}$ be convex and non-decreasing. Show that the composition $f \circ g(\mathbf{x}) = f(g(\mathbf{x}))$ of the functions f and g is convex.

(c) Show : The function $f(x) = e^{\|\mathbf{x}\|}$ is convex on $\mathbb{R}^n$ (for any norm $\|\mathbf{x}\|$ on $\mathbb{R}^n$).

Ex. 4

(a) Let $f : (a, b) \to \mathbb{R}$ be a convex function. Show for all $x \in (a, b)$:

$$\partial f(x) = \{d \in \mathbb{R} \mid f'_-(x) \le d \le f'_+(x)\}.$$

(b) Determine the subdifferentials for the function $f(x) = |x^2 - 1|$ at the points $x_0 = 0$, $x_1 = 1$, and $x_2 = 4$. (Is the function f convex?).

Ex. 5 Given the function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(\mathbf{x}) = x_1^3 + e^{3x_2} - 3x_1 e^{x_2} \ .$$

(a) Find the critical points (i.e. points satisfying $\nabla f(\mathbf{x}) = \mathbf{0}^T$, $\mathbf{x} = (x_1, x_2)$) of the function f and determine the local minimizers.

(b) Does there exist a global minimizer or a global maximizer of f on $\mathbb{R}^n$?

(c) Suppose we apply the steepest descent method to f. What can you say about the (local) convergence properties. (Quadratic or linear convergent? Give an estimate for the convergence factor.)

Points: 36+4 =40

Ex. 1	a	:	3 pt.
	b	:	4 pt.
Ex. 2	a	:	3 pt.
	b	:	4 pt.
Ex. 3	a	:	2 pt.
	b	:	4 pt.
	c	:	1 pt.
Ex. 4	a	:	5 pt.
	b	:	2 pt.
Ex. 5	a	:	4 pt.
	b	:	2 pt.
	c	:	2 pt.

The scipt ’Mathematical Programming I’
may be used during the examination. Good luck!