Ex. 4 Consider the problem (in connection with the design of a cylindrical can with height h, radius r and volume at least 2π such that the total surface area is minimal):

$$(P): \quad \min\ f(h,r) := 2\pi(r^2 + rh) \quad \text{s.t.}\ -\pi r^2 h \le -2\pi, \quad (\text{and } h > 0, r > 0)$$

(a) Compute a (the) solution $(\overline{h}, \overline{r})$ of the KKT conditions of (P). Show that (P) is not a convex optimization problem.

(b) Show that the solution $(\overline{h}, \overline{r})$ in (a) is a local minimizer. Why is it the unique global solution? *Hint: Use the sufficient optimality conditions*

Solution:

(a) We first note that the functions $f(h,r) = 2\pi(r^2 + rh)$ and $g(h,r) := -\pi r^2 h + 2\pi$ are not convex (for $h > 0$). For the objective function f,e.g., this follows by:

$$\nabla f = 2\pi\begin{pmatrix} r \\ 2r+h \end{pmatrix}, \quad \nabla^2 f = 2\pi\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and thus:} \quad \det \nabla^2 f < 0$$

We now consider the KKT condition: $(\nabla f = -\mu\nabla g, g \le 0, \mu \cdot g = 0)$

So consider: $2\pi\binom{r}{2r+h} = \mu\pi\binom{r^2}{2rh}$ $(\star)$:

Case $\mu = 0$: leads to $2\pi\binom{r}{2r+h} = 0$ with solution $(h,r) = (0,0)$ which is not feasible.

Case $\mu > 0$ and thus $\pi r^2 h = 2\pi$:

The 2 equations in $(\star)$ lead to $\mu = 2/r$ and then $2(2r+h) = \frac{2}{r}2rh$ or $h = 2r$. By using the (active) constraint we find $\pi r^2 h = 2\pi r^3 = 2\pi$ with solution $r = 1$. So the unique KKT solution is given by $(\overline{h}, \overline{r}) = (2,1), \overline{\mu} = 2$.

(b) (We apply the second order sufficient conditions to the nonconvex program (P)).

So we will show (for the cone of critical directions $C(\overline{h}, \overline{r})$):

$$d^T \nabla^2 L(\overline{h}, \overline{r}, \overline{\mu}) d > 0 \quad \forall d \in C(\overline{h}, \overline{r}) \setminus \{0\} \quad (\star\star)$$

We compute

$$\nabla f(\overline{h}, \overline{r}) = 2\pi\begin{pmatrix} 1 \\ 4 \end{pmatrix}, \nabla g(\overline{h}, \overline{r}) = -\pi\begin{pmatrix} 1 \\ 4 \end{pmatrix}, \ \nabla^2 L(\overline{h}, \overline{r}, \overline{\mu}) = 2\pi\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} + 2(-\pi)\begin{pmatrix} 0 & 2 \\ 2 & 4 \end{pmatrix} = -2\pi\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$$

and

$$\begin{aligned} C(\overline{h}, \overline{r}) &= \{d \in \mathbb{R}^2 \mid \nabla f(\overline{h}, \overline{r})^T d \le 0, \nabla g(\overline{h}, \overline{r})^T d \le 0\} \\ &= \{d \in \mathbb{R}^2 \mid \begin{pmatrix} 1 \\ 4 \end{pmatrix}^T d \le 0, \ -\begin{pmatrix} 1 \\ 4 \end{pmatrix}^T d \le 0\} \\ &= \{\lambda\begin{pmatrix} -4 \\ 1 \end{pmatrix} \mid \lambda \in \mathbb{R}\} \end{aligned}$$

For $d = \lambda(-4,1)^T, \lambda \ne 0$ we obtain (see $(\star\star)$):

$$\lambda(-4,1)(-2\pi)\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}\lambda\begin{pmatrix} -4 \\ 1 \end{pmatrix} = \ldots = 2\lambda^2\pi 6 > 0 \quad \forall \lambda \ne 0\,.$$

So $(\overline{h}, \overline{r}) = (2,1)$ is a local minimizer.

It is the unique (global) minimizer since the point is the only KKT point.

Note that since the linear independency constraint qualification holds (for $r, h > 0$) any local minimizer must be a KKT point. Also note that for feasible $\|(h,r)\| \to \infty$ also $f \to \infty$ holds. (*To show the latter fact is technically "involved" and was not expected to be done.*)