Test exam: Continuous Optimisation 2014

3TU- and LNMB-course, Utrecht.

Monday 8^{th} December 2014

1. Let $f : \mathbb{R}^m \to \mathbb{R}$ be a convex function $f(y)$ on $\mathbb{R}^m$ and let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ be given.
   (a) Show that the function $g(x) := f(Ax + b)$ is a convex function of x on $\mathbb{R}^n$. [3 points]
   (b) Suppose that f is strictly convex. Show that then $g(x) := f(Ax+b)$ is strictly convex if and only if A has (full) rank n. [4 points]
   Hint: Recall that f is strictly convex if for any $y_1 \neq y_2$, $0 < \lambda < 1$ it holds: $f(\lambda y_1 + (1-\lambda)y_2) < \lambda f(y_1) + (1-\lambda)f(y_2)$.

2. For given $S \subset \mathbb{R}^n$ we define the convex hull conv(S) by

$$\operatorname{conv}(S) = \left\{ x = \sum_{i=1}^{m} \lambda_i x_i \;\middle|\; \sum_{i=1}^{m} \lambda_i = 1;\ x_i \in S, \lambda_i \geq 0\ \forall i;\ m \in \mathbb{N} \right\}$$

   Show that conv(S) is the smallest convex set containing S:
   (a) Show that the set conv(S) is convex. [3 points]
   (b) Show that for any convex set C containing S we must have conv$(S) \subset C$. [3 points]
   (*Hint: You may use without proof any Lemma, Theorem etc. from the course*)

3. Consider with $0 \neq c \in \mathbb{R}^n$ the program:

$$(P) \qquad \min_{x\in\mathbb{R}^n} c^T x \quad \text{s.t.} \quad x^T x \leq 1 \,.$$

   (a) Show that $\overline{x} = -\frac{c}{\|c\|}$ is the minimizer of (P) with minimum value $v(P) = -\|c\|$. [2 points]
   ($\|x\|$ means here the Euclidian norm.)
   (b) Compute the solution $\overline{y}$ of the Lagrangean dual (D) of (P). Show in this way that for the optimal values strong duality holds, i.e., $v(D) = v(P)$. [4 points]

4. Consider the problem (in connection with the design of a cylindrical can with height h, radius r and volume at least 2π such that the total surface area is minimal):

$$(P): \quad \min\ f(h,r) := 2\pi(r^2 + rh) \quad \text{s.t.} \quad -\pi r^2 h \le -2\pi, \quad (\text{and } h > 0, r > 0)$$

   (a) Compute a (the) solution $(\overline{h}, \overline{r})$ of the KKT conditions of (P). Show that (P) is not a convex optimization problem. [4 points]

   (b) Show that the solution $(\overline{h}, \overline{r})$ in (a) is a local minimizer. Why is it the unique global solution? [3 points]
   Hint: Use the sufficient optimality conditions

5. Consider the closed set

$$\mathcal{K} = \{\mathbf{x} \in \mathbb{R}^2 \mid x_1 + 2x_2 \ge 0 \text{ and } 3x_1 + x_2 \ge 0\}$$

   (a) Prove that $\mathcal{K}$ is a convex cone. [2 points]

   (b) Prove that $\mathcal{K}$ is full-dimensional. [1 point]

   (c) Prove that $\mathcal{K}$ is pointed. [2 points]

   (d) Find the dual cone to $\mathcal{K}$. [1 point]

6. We will consider bounds to the optimal value of the following problem:

$$\begin{aligned} \min_{\mathbf{x}} \quad & 5x_1^2 - 4x_1x_2 - 2x_1 + x_2^2 + 2 \\ \text{s.t.} \quad & \mathbf{x} \in \mathbb{R}^2. \end{aligned} \tag{A}$$

   (a) Give an upper bound on the optimal value of problem (A). [1 point]

   (b) Formulate a sum-of-squares optimisation problem to give a lower bound on the optimal value of problem (A). [1 point]

   (c) For fixed $u \in \mathbb{R}$, consider the polynomial $f(\mathbf{x}) = 5x_1^2 - 4x_1x_2 - 2x_1 + x_2^2 + u$. Write the constraint that f is a sum-of-squares polynomial explicitly as a positive semidefinite constraint. [2 points]

7. (Automatic additional points) [4 points]

Question:	1	2	3	4	5	6	7	Total
Points:	7	6	6	7	6	4	4	40

A copy of the lecture-sheets may be used during the examination.
Good luck!