Some properties of an upper bound for μ

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Abstract— **A convex upper bound of the mixed structured singular value μ is analyzed. The upper bound is based on a multiplier method. It is simple, it can exploit low-rank properties and it is shown to be less conservative than the well-known (D,G)-scaling. A direct relationship with (D,G)-scaling is given. The upper bound can be modified to one that is continuous with an explicit Lipschitz constant.**

Keywords— **Mixed structured singular values, linear matrix inequalities, multipliers.**

I. Introduction

One of the nagging problems of robust control is to determine whether or not a system remains stable and retains satisfactory performance qualities under variations and uncertainties of some sort. In an attempt to get a handle on the problem, Doyle [8] introduced in 1982 the *structured singular value*. The definition of the structured singular value, or μ for short, is such that robust stability/robust performance of the control problem is equivalent to μ being less than 1. Originally μ was defined for problems where the performance is measured by an $\mathcal{H}_\infty$-norm and the uncertainties come in the form of structured complex matrices, such as due to neglected dynamics. It was soon realized, however, that the definition μ could be extended to handle real parametric uncertainties as well. This extension of μ is known as the *mixed* structured singular value [9].

Unfortunately, in its full generality the test "$\mu < 1$?" is NP-hard [5], [17], and, as such any attempt to develop polynomial-time algorithms for the general case seems futile. At present efficient algorithms work for special cases only, or, if they do work for the general case then they provide only *sufficient* conditions (see Young, Newlin and Doyle [20] and the survey papers by Packard and Doyle [16] and Barmish and Kang [2]). The role of upper bounds of μ is that they provide such sufficient conditions. Of the computable upper bounds of μ that have been reported the best at present that can handle the general μ problem are those by Fan, Tits and Doyle [9] and Fu and Barabanov [11].

In this note we examine an upper bound of μ based on what is called the multiplier method. The idea of the multiplier method is that nonsingularity of a family of matrices $\mathcal{T}$ is ensured if for some matrix C we have that $CT + T^*C^* > 0$ for all $T \in \mathcal{T}$. Note that the condition $CT + T^*C^* > 0$ is convex in the variable C. The idea is not really new, although its application for the general μ problem was only recently realized by Fu and Barabanov [11]. The multiplier approach is extremely simple and, as Fu and Barabanov showed, it is readily translated using the S-procedure into linear matrix inequalities (LMIs) and generalized eigenvalue problems (GEVPs). LMIs and GEVPs are nowadays efficient to solve (see Nesterov and Nemirovsky [15] and Boyd et.al. [4]). The simplicity of the multiplier method may give the impression that the upper bounds of μ derived from it are conservative. Interestingly, however, the best advocated upper bound of μ by Fan, Tits and Doyle [9] can be seen as an example of the multiplier method and turns out to be more conservative than—or, at best, equal to—any of the upper bounds derived in [11] from the multiplier method.

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We collect in this note some general properties of the multiplier method and the upper bound of μ it induces. Section II introduces notation and some well established results. Section III discusses the multiplier method. Firstly we show that the upper bound of Fan, Tits and Doyle [9] is more conservative than the general multiplier method. This fact was realized already by Fu and Barabanov [11], but here we provide an explicit relationship. In Section IV we show that many variations of the multiplier method are equivalent. This result allows us to exploit low-rank conditions to reduce the computational cost without any conservatism. We briefly comment on the lack of continuity of μ and its upper bounds, and how the multiplier method may be modified to recover continuity. Owing to the simplicity of the method we obtain Lipschitz continuity with an explicit Lipschitz constant.

II. Preliminaries

In this section we summarize material that we need later. The mixed structured singular value is defined and we briefly review (D,G)-scaling and the generalized eigenvalue problem.

The norm $\|T\|$ of a matrix T is in this note the spectral norm. T^* is the complex conjugate transpose of T and the *Hermitian part* He T of T is defined as

$$\operatorname{He} T = \frac{1}{2}(T + T^*).$$

Given a subset $\mathbb{X}$ of $\mathbb{C}^{n\times n}$ we use $\mathcal{B}_\mathbb{X}$ to denote the unit ball in $\mathbb{X}$:

$$\mathcal{B}_\mathbb{X} = \{\, \Delta \in \mathbb{X} \;:\; \|\Delta\| \leq 1 \,\}.$$

Given $\mathbb{X}$ the *mixed structured singular value* of M, denoted by $\mu_\mathbb{X}(M)$, is defined as the infimal value of $\alpha > 0$ for which all elements of the set $I - \frac{1}{\alpha}\mathcal{B}_\mathbb{X} M$ are nonsingular. Obviously $\mu_\mathbb{X}(M)$ depends on the "structure" $\mathbb{X}$. Invariably, the structures considered are of the form

$$\begin{aligned} \mathbb{X} \;=\; \operatorname{diag}\,(&\mathbb{R}I_{k_1}, \ldots, \mathbb{R}I_{k_{m_r}}, \\ &\mathbb{C}I_{l_1}, \ldots, \mathbb{C}I_{l_{m_c}}, \\ &\mathbb{C}^{f_1\times f_1}, \ldots, \mathbb{C}^{f_{m_C}\times f_{m_C}}), \end{aligned} \tag{1}$$

where m_r, m_c and m_C are the number of repeated real scalar blocks, repeated complex scalar blocks and full complex blocks, respectively.

A. (D,G)-scaling, LMIs and GEVPs

Let $\mathbb{H}^q$ denote the set of $q\times q$ Hermitian matrices and denote its subset of positive definite elements by $\mathbb{P}^q$. Given the structure $\mathbb{X}$ of (1), the sets $\mathcal{D}_\mathbb{X}$ and $\mathcal{G}_\mathbb{X}$ are defined as

$$\begin{aligned} \mathcal{D}_\mathbb{X} \;=\; \operatorname{diag}\,(&\mathbb{P}^{k_1}, \ldots, \mathbb{P}^{k_{m_r}}, \\ &\mathbb{P}^{l_1}, \ldots, \mathbb{P}^{l_{m_c}}, \\ &\mathbb{P} I_{f_1}, \ldots, \mathbb{P} I_{f_{m_C}}), \\ \mathcal{G}_\mathbb{X} \;=\; \operatorname{diag}\,(&\mathbb{H}^{k_1}, \ldots, \mathbb{H}^{k_{m_r}}, \\ &0_{l_1\times l_1}, \ldots, 0_{l_{m_c}\times l_{m_c}}, \\ &0_{f_1\times f_1}, \ldots, 0_{f_{m_C}\times f_{m_C}}). \end{aligned}$$

Note that for all $\Delta \in \mathbb{X}$, $D \in \mathcal{D}_\mathbb{X}$ and $G \in \mathcal{G}_\mathbb{X}$ we have that $D\Delta = \Delta D$; $D^{1/2}\Delta = \Delta D^{1/2}$; $\Delta G = \Delta^* G = G\Delta$. Given M we define $\Phi_\alpha(D,G)$ as

$$\Phi_\alpha(D,G) = M^*DM + j(GM - M^*G) - \alpha^2 D.$$

This notation is a bit different from that of [9]. Fan, Tits and Doyle [9] showed that $\mu_\mathbb{X}(M) < \alpha$ if $\alpha > 0$ and $\Phi_\alpha(D,G) < 0$

for some $D \in \mathcal{D}_{\mathbb{X}}$ and $G \in \mathcal{G}_{\mathbb{X}}$. So, in particular the α that solves the constrained minimization problem

$$\begin{array}{ll} \text{minimize} & \alpha \\ \text{subject to} & \Phi_\alpha(D,G) < 0,\ \alpha > 0,\ D \in \mathcal{D}_{\mathbb{X}},\ G \in \mathcal{G}_{\mathbb{X}} \end{array} \tag{2}$$

is an upper bound of $\mu_{\mathbb{X}}(M)$. This minimum α is denoted as $\nu_{\mathbb{X}}(M)$. The minimization problem (2) is an example of a *generalized eigenvalue problem* (GEVP). A GEVP is a minimization problem of the form

$$\begin{array}{ll} \underset{\alpha\in\mathbb{R},x\in\mathbb{R}^{q\times p}}{\text{minimize}} & \alpha \\ \text{subject to} & \alpha B(x) - A(x) > 0,\ B(x) > 0,\ E(x) > 0 \end{array}$$

where $A(x)$, $B(x)$ and $E(x)$ are square symmetric matrices depending affinely on the variable $x \in \mathbb{R}^{q\times p}$. For GEVPs efficient polynomial-time algorithms exist [15], [4], [12]. A *linear matrix inequality* (LMI) in the variable $x \in \mathbb{R}^{p\times q}$ is an inequality $E(x) > 0$ where $E(x)$ is a square symmetric matrix affine in x.

III. The multiplier method

In this section we examine an upper bound of $\mu_{\mathbb{X}}$ that relies on the multiplier method as developed by Fu and Barabanov [11]. In particular we provide a connection with (D,G)-scaling. At the basis of the multiplier method lies the trivial observation that

a square matrix T is nonsingular if there is a square matrix C such that He $CT > 0$.

This is easy to see: If T is singular with $Tv = 0$, $v \neq 0$ we get for any C that $v^*(CT + T^*C^*)v = 0$, contradicting that $CT + T^*C^* = 2\text{He}\ CT > 0$. The matrix C is called *multiplier* because of its links with the Popov multiplier, although it is not quite the same as that of Popov [18]. Popov style multipliers, in the modern setting, appear in the work of Safonov and others—see [7], [14]—and they are in way equivalent to (D,G)-scaling (see [1]). The approach we use here can be traced back to an idea by Brockett and Willems [6] from 1965. The following result is immediate.

Theorem III.1: $\mu_{\mathbb{X}}(M) < \alpha$ for an $n\times n$ matrix M if there is a single multiplier $C \in \mathbb{C}^{n\times n}$ such that

$$\text{He}\ C(I_n - \tfrac{1}{\alpha}\mathcal{B}_{\mathbb{X}}M) > 0. \tag{3}$$

By (3) is meant that all elements of the set $C(I_n - \frac{1}{\alpha}\mathcal{B}_{\mathbb{X}}M)$ have positive definite Hermitian part.

If C satisfies (3) we say that C is a *feasible multiplier* for the set $I_n - \frac{1}{\alpha}\mathcal{B}_{\mathbb{X}}M$. The infimal α for which such a feasible multiplier C can be found is thus an upper bound of $\mu_{\mathbb{X}}(M)$. For later reference we call this upper bound $\widehat{\mu}_{\mathbb{X}}(M)$:

$$\widehat{\mu}_{\mathbb{X}}(M) := \inf\{\alpha > 0 : \exists C \text{ s.t. He}\ C(I_n - \tfrac{1}{\alpha}\mathcal{B}_{\mathbb{X}}M) > 0\}. \tag{4}$$

Even though the inequality (3) is convex in C, computation of $\widehat{\mu}_{\mathbb{X}}$ is generally not possible due to the infinite family $\frac{1}{\alpha}\mathcal{B}_{\mathbb{X}}$. There is an exception. If the structure $\mathbb{X}$ has real components only, that is,

$$\mathbb{X} = \text{diag}\,(\mathbb{R}I_{k_1}, \dots, \mathbb{R}I_{k_{m_r}}), \tag{5}$$

then by convexity (3) holds for multiplier C iff the finitely many LMIs

$$\text{He}\ C(I_n - \Delta M) > 0\ \ \forall \Delta = \frac{1}{\alpha}\text{diag}\,(\pm I_{k_1}, \dots, \pm I_{k_{m_r}}), \tag{6}$$

are satisfied. Minimizing $\alpha > 0$ over all C subject to (6) is a GEVP. Thus it seems that computation of $\widehat{\mu}_{\mathbb{X}}$ is well suited for the real case. This is in contrast with (D,G)-scaling which is primarily useful for the non-real case. In any case, real or not, the following holds.

Theorem III.2: For every structure $\mathbb{X}$ and matrix M we have

$$\mu_{\mathbb{X}}(M) \le \widehat{\mu}_{\mathbb{X}}(M) \le \nu_{\mathbb{X}}(M).$$

In particular, if $\Phi_\alpha(D,G) < 0$ with $D \in \mathcal{D}_{\mathbb{X}}$, $G \in \mathcal{G}_{\mathbb{X}}$ then $C = \alpha^2 D + jM^*G$ satisfies (3).

Proof: Let $C = \alpha^2 D + jM^*G$. It is readily verified that for every $\Delta \in \mathbb{X}$ we have

$$\begin{aligned} 2\text{He}\ C(I-\Delta M) &= C(I-\Delta M) + (I-\Delta M)^*C^* = \\ &-\Phi_\alpha(D,G)\ +\ M^*D^{1/2}(I-\alpha^2\Delta^*\Delta)D^{1/2}M \\ &+\alpha^2(D^{1/2}-\Delta D^{1/2}M)^*(D^{1/2}-\Delta D^{1/2}M). \end{aligned} \tag{7}$$

Therefore He $C(I-\Delta M) > 0$ for all $\Delta \in \frac{1}{\alpha}\mathcal{B}_{\mathbb{X}}$ if $\Phi_\alpha(D,G) < 0$.

In (7) we used the fact that for every $\Delta \in \mathbb{X}$ we have $D^{1/2}\Delta = \Delta D^{1/2}$ and $G\Delta = \Delta^* G$. ∎

So (D,G)-scaling can be interpreted as being a restrictive special case of the multiplier method. On the other hand, while $\nu_{\mathbb{X}}$ is efficient to compute, computation of $\widehat{\mu}_{\mathbb{X}}$ is generally possible only for real-valued structures (5) and even then care has to be taken since the computational cost grows exponentially with the number of real blocks m_r. In [11] it is explained how to combine the advantages of the multiplier method with the advantages of (D,G)-scaling. Loosely speaking the idea is to apply the multiplier method on the first, say, 5 real blocks, and to apply (D,G)-scaling on the *remaining* real blocks (and complex blocks). As shown in [11] these combined computable upper bounds of $\mu_{\mathbb{X}}$ are less conservative than $\nu_{\mathbb{X}}$. It remains to be seen how these upper bounds compare to (D,G)-scaling in combination with a gridding of the real parameter space, which is an alternative method to reduce conservatism of the upper bound $\nu_{\mathbb{X}}$. We end this section with an example that shows that $\widehat{\mu}_{\mathbb{X}}$ is generally strictly less than $\nu_{\mathbb{X}}$.

Example III.3: We derive in this example the values of $\mu_{\mathbb{X}}(M)$, $\nu_{\mathbb{X}}(M)$ and $\widehat{\mu}_{\mathbb{X}}(M)$ for the case that

$$\mathbb{X} = \begin{bmatrix} \mathbb{R} & 0 \\ 0 & \mathbb{R} \end{bmatrix} \quad\text{and}\quad M = \begin{bmatrix} 0 & 1 \\ j & 0 \end{bmatrix}.$$

For all $\Delta = \text{diag}\,(\delta_1, \delta_2) \in \mathbb{X}$ the determinant $\det(I_2 - \Delta M)$ equals $1 - j\delta_1\delta_2$. This is nonzero irrespective of the values of $\delta_1 \in \mathbb{R}$ and $\delta_2 \in \mathbb{R}$, hence $\mu_{\mathbb{X}}(M) = 0$.

For this example $\mathcal{D}_{\mathbb{X}}$ is the set of diagonal positive definite matrices, and $\mathcal{G}_{\mathbb{X}}$ is the set of real diagonal matrices. With $D \in \mathcal{D}_{\mathbb{X}}$ and $G \in \mathcal{G}_{\mathbb{X}}$ written as $D = \text{diag}\,(d_1, d_2)$ and $G = \text{diag}\,(g_1, g_2)$ we get that

$$\begin{aligned} \Phi_\alpha(D,G) &= M^*DM + j(GM - M^*G) - \alpha^2 D \\ &= \begin{bmatrix} d_2 - \alpha^2 d_1 & jg_1 - g_2 \\ -jg_1 - g_2 & d_1 - \alpha^2 d_2 \end{bmatrix}. \end{aligned} \tag{8}$$

Based on the diagonal elements of (8) it can be seen that $\Phi_\alpha(D,G) < 0$ for some $D \in \mathcal{D}_{\mathbb{X}}$ only if $\alpha^2 > 1$. On the other hand, (8) is negative definite for every $\alpha^2 > 1$ if we choose $d_1 = d_2 = 1$ and $g_1 = g_2 = 0$. Hence $\nu_{\mathbb{X}}(M) = 1$.

For $C = I_2$ and $\Delta = \text{diag}\,(\delta_1, \delta_2) \in \mathbb{X}$ we have that

$$\text{He}\ C(I_2 - \Delta M) = \begin{bmatrix} 1 & (j\delta_2 - \delta_1)/2 \\ (-j\delta_2 - \delta_1)/2 & 1 \end{bmatrix}.$$

This is positive definite for all real δ_i satisfying $|\delta_i| < \sqrt{2}$. Hence $\widehat{\mu}_{\mathbb{X}}(M) \le 1/\sqrt{2}$. A further technical analysis shows that $\widehat{\mu}_{\mathbb{X}}(M)$ is in fact equal to $1/\sqrt{2}$. So, for this example none of the bounds are the same: $0 = \mu_{\mathbb{X}}(M) < \widehat{\mu}_{\mathbb{X}}(M) = 1/\sqrt{2} < \nu_{\mathbb{X}}(M) = 1$.

IV. Equivalent multipliers

The multiplier method has many variations. For example, instead of trying to find a multiplier C such that (3) holds, it might perhaps be advantageous to try to find a scalar c for which

$$\text{He } c \det(I - \Delta M) > 0 \quad \text{for all } \Delta \in \tfrac{1}{\alpha}\mathcal{B}_{\mathbb{X}}. \tag{9}$$

If such a c exist then again $\mu_{\mathbb{X}}(M) < \alpha$. In this section we discuss a few of these variations. We show that certain variations are equivalent in that they induce the same upper bound of $\mu_{\mathbb{X}}$. The class of equivalent multiplier methods includes one that exploits low-rank properties of M, but it does not include the one in (9).

Numerous robust stability problems require computation of $\mu_{\mathbb{X}}(M)$ in which the matrix M has low rank. In fact, M is often constructed from tall matrices P and Q as

$$M = QP^*,$$

with P and Q readily given by the problem at hand:

Example IV.1: Let $\{A_0, \dots, A_m\}$ be a collection of $q \times q$ matrices and let A be any $q \times q$ matrix of the form

$$A = A_0 + \delta_1 A_1 + \cdots + \delta_m A_m, \quad -1 < \delta_i < 1.$$

It is well known that the uncertain system $\dot{x} = Ax$ is asymptotically stable for all such matrices A iff A_0 has all its eigenvalues in the open left-half plane and for all frequencies $\omega \in \mathbb{R}$ we have that

$$I_q - \sum_{i=1}^{m} \delta_i A_i (j\omega I_q - A_0)^{-1} \quad \text{is nonsingular } \forall |\delta_i| < 1. \tag{10}$$

The above condition (10) is a low-rank μ problem. To see this define $\Delta = \operatorname{diag}(\delta_1 I_q, \dots, \delta_m I_q)$ and define the tall matrices P and Q_ω as

$$P = \begin{bmatrix} I_q \\ \vdots \\ I_q \end{bmatrix}, \quad Q_\omega = \begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix} (j\omega I_q - A_0)^{-1}.$$

This way the matrix $I_q - \sum_{i=1}^{m} \delta_i A_i (j\omega I_q - A_0)^{-1}$ of (10) can be written as $I_q - P^*\Delta Q_\omega$. By the determinant rule $\det(I - AB) = \det(I - BA)$ it now follows that (10) holds for all frequencies iff with respect to structure $\mathbb{X} = \operatorname{diag}(\mathbb{R}I_q, \dots, \mathbb{R}I_q)$ we have

$$\mu_{\mathbb{X}}(Q_\omega P^*) < 1$$

for all frequencies $\omega \in \mathbb{R}$. The matrix $M := Q_\omega P^*$ is $qm \times qm$ but only has rank q.

In general, robust nonsingularity of the $n \times n$ set

$$I_n - \tfrac{1}{\alpha}\mathcal{B}_{\mathbb{X}} QP^*$$

is the same as robust nonsingularity of the "smaller" $q \times q$ set

$$I_q - P^* \tfrac{1}{\alpha}\mathcal{B}_{\mathbb{X}} Q.$$

The obvious question is whether or not the multiplier method applied to the smaller $I_q - P^* \frac{1}{\alpha}\mathcal{B}_{\mathbb{X}} Q$ provides a more conservative upper bound of $\mu_{\mathbb{X}}(QP^*)$ than $\widehat{\mu}_{\mathbb{X}}(QP^*)$. They are equivalent:

Theorem IV.2: Let $M = QP^*$. The following four upper bounds of $\mu_{\mathbb{X}}(M)$ are the same.

$$\begin{aligned}
\widehat{\mu}_{\mathbb{X}}(M) &:= \inf\{\alpha > 0 \ : \exists C \text{ s.t. He } C(I_n - \tfrac{1}{\alpha}\mathcal{B}_{\mathbb{X}} M) > 0\,\}, \\
\widehat{\mu}_{\mathbb{X},2}(P,Q) &:= \inf\{\alpha > 0 \ : \exists C_2 \text{ s.t. He } C_2(I_q - P^* \tfrac{1}{\alpha}\mathcal{B}_{\mathbb{X}} Q) > 0\,\}, \\
\widehat{\mu}_{\mathbb{X},3}(P,Q) &:= \inf\{\alpha > 0 \ : \exists C_3 \text{ s.t. He } (I_q - P^* \tfrac{1}{\alpha}\mathcal{B}_{\mathbb{X}} Q)C_3 > 0\,\}, \\
\widehat{\mu}_{\mathbb{X},4} &:= \inf\{\alpha > 0 \ : \exists C_4 \text{ s.t. He } (I_n - \tfrac{1}{\alpha}\mathcal{B}_{\mathbb{X}} M)C_4 > 0\,\}.
\end{aligned}$$

Proof: We prove in four steps that $\widehat{\mu}_{\mathbb{X}} \le \widehat{\mu}_{\mathbb{X},2} = \widehat{\mu}_{\mathbb{X},3} \le \widehat{\mu}_{\mathbb{X},4} = \widehat{\mu}_{\mathbb{X}}$, which shows they are all equal.

($\widehat{\mu}_{\mathbb{X}} \le \widehat{\mu}_{\mathbb{X},2}$): Suppose $\alpha > \widehat{\mu}_{\mathbb{X},2}$ and that C_2 satisfies He $C_2(I_q - P^* \frac{1}{\alpha}\mathcal{B}_{\mathbb{X}} Q) > 0$. Consider $C := PC_2P^* + \varepsilon EE^*$ with E and ε yet to be determined. Then

$$\begin{aligned}
C(I_n - \Delta M) &= (PC_2P^* + \varepsilon EE^*)(I_n - \Delta QP^*) \\
&= \begin{bmatrix} P & E \end{bmatrix} \begin{bmatrix} C_2(I_q - P^*\Delta Q) & 0 \\ -\varepsilon E^*\Delta Q & \varepsilon I \end{bmatrix} \begin{bmatrix} P^* \\ E^* \end{bmatrix}.
\end{aligned}$$

Therefore its Hermitian part equals

$$\begin{aligned}
&\text{He } C(I_n - \Delta M) = \\
&\begin{bmatrix} P & E \end{bmatrix} \begin{bmatrix} \text{He } C_2(I_q - P^*\Delta Q) & -\frac{\varepsilon}{2} Q^*\Delta^* E \\ -\frac{\varepsilon}{2} E^*\Delta Q & \varepsilon I \end{bmatrix} \begin{bmatrix} P^* \\ E^* \end{bmatrix}.
\end{aligned} \tag{11}$$

For any E for which $[\,P \quad E\,]$ has full row rank there is a small enough $\varepsilon > 0$ such that (11) is > 0 for all $\Delta \in \frac{1}{\alpha}\mathcal{B}_{\mathbb{X}}$. Hence $\widehat{\mu}_{\mathbb{X}} < \alpha$. Since this can be done for any $\alpha > \widehat{\mu}_{\mathbb{X},2}$ we have that $\widehat{\mu}_{\mathbb{X}} \le \widehat{\mu}_{\mathbb{X},2}$.

($\widehat{\mu}_{\mathbb{X},2} = \widehat{\mu}_{\mathbb{X},3}$): Take $C_3 = C_2^{-*}$.

($\widehat{\mu}_{\mathbb{X},3} \le \widehat{\mu}_{\mathbb{X},4}$): Suppose $\alpha > \widehat{\mu}_{\mathbb{X},4}$ and that C_4 satisfies He $(I_n - \frac{1}{\alpha}\mathcal{B}_{\mathbb{X}} M)C_4 > 0$. Consider $C_3 := P^*C_4P + \varepsilon E^*E$ with E and ε yet to be determined. Then

$$\begin{aligned}
(I_q - P^*\Delta Q)C_3 &= (I_q - P^*\Delta Q)(P^*C_4P + \varepsilon E^*E) \\
&= \begin{bmatrix} P^* & E^* \end{bmatrix} \begin{bmatrix} (I_n - \Delta M)C_4 & -\varepsilon \Delta Q E^* \\ 0 & \varepsilon I \end{bmatrix} \begin{bmatrix} P \\ E \end{bmatrix}.
\end{aligned}$$

Therefore its Hermitian part equals

$$\begin{aligned}
&\text{He } (I_q - P^*\Delta Q)C_3 = \\
&\begin{bmatrix} P^* & E^* \end{bmatrix} \begin{bmatrix} \text{He } (I_n - \Delta M)C_4 & -\frac{\varepsilon}{2}\Delta Q E^* \\ -\frac{\varepsilon}{2} E Q^*\Delta & \varepsilon I \end{bmatrix} \begin{bmatrix} P \\ E \end{bmatrix}.
\end{aligned} \tag{12}$$

For any E for which $[\,P^* \quad E^*\,]$ has full row rank there is a small enough $\varepsilon > 0$ such that (12) is > 0 for all $\Delta \in \frac{1}{\alpha}\mathcal{B}_{\mathbb{X}}$. Hence $\widehat{\mu}_{\mathbb{X},3} < \alpha$. Since this can be done for any $\alpha > \widehat{\mu}_{\mathbb{X},4}$ we have that $\widehat{\mu}_{\mathbb{X},3} \le \widehat{\mu}_{\mathbb{X},4}$.

($\widehat{\mu}_{\mathbb{X},4} = \widehat{\mu}_{\mathbb{X}}$): Take $C = C_4^{-*}$. ■

The equivalence of $\widehat{\mu}_{\mathbb{X}}$ and $\widehat{\mu}_{\mathbb{X},4}$ shows that flipping the multiplier to the other side of the nonsingularity set has no effect on the upper bound it induces. A further variation of the multiplier method which is easily verified is that $\widehat{\mu}_{\mathbb{X}}(QR^{-1}P^*)$ equals

$$\inf\{\alpha > 0 \ : \exists C_5 \text{ such that He } C_5(R - P^* \tfrac{1}{\alpha}\mathcal{B}_{\mathbb{X}} Q) > 0\,\}.$$

Example IV.3: Consider Example IV.1. For a fixed frequency ω, the upper bound $\widehat{\mu}_{\mathbb{X}}(Q_\omega P^*)$—which involves a $qm \times qm$ multiplier—is the same as the infimal $\alpha > 0$ for which a $C_5 \in \mathbb{C}^{q \times q}$ exists such that

$$\text{He } C_5[(j\omega I_q - A_0) - \sum_{i=1}^{m} \delta_i A_i] > 0 \quad \text{for all } \delta_i \in [-\tfrac{1}{\alpha}, \tfrac{1}{\alpha}].$$

The equivalence of the various multiplier upper bounds generally breaks down if $I_n - \Delta M$ or $I_q - P^*\Delta Q$ is replaced with one that no longer depends affinely on Δ. For example, we know that $I_n - \Delta M$ is nonsingular iff $\det(I_n - \Delta M)$ is nonzero. However the existence of a multiplier C for $I_n - \Delta M$ satisfying (3) does not necessarily imply that a scalar c exists such that

$$\text{He } c \det(I - \tfrac{1}{\alpha}\mathcal{B}_{\mathbb{X}} M) > 0. \tag{13}$$

As an example, suppose we have

$$\mathbb{X} = \begin{bmatrix} \mathbb{R} & 0 \\ 0 & \mathbb{R} \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 2j & 0 \\ 0 & 2j \end{bmatrix}.$$

Then $\mu_{\mathbb{X}}(M) = \widehat{\mu}_{\mathbb{X}}(M) = 0$ because for any $\alpha > 0$ we have He $(I_2 - \frac{1}{\alpha}\mathcal{B}_{\mathbb{X}}M) = \{I_2\} > 0$. However, for, say, $\alpha = 1$ we can not find a c that satisfies (13) because for the four Δs

$$\Delta = \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix} \in \tfrac{1}{\alpha}\mathcal{B}_{\mathbb{X}}$$

the determinant $\det(I - \Delta M) = (1 \pm 2j)(1 \pm 2j)$ takes the values $-3 \pm 4j$ and 5 (twice) and these do not lie in one half-space $\{\, s : \theta + \pi/2 > \arg s > \theta - \pi/2 \,\}$.

It is an important asset of the multiplier method that it can exploit the low rank properties of M. The reduction in size when going from the $n \times n$ matrix $I_n - \Delta M$ to the equivalent $q \times q$ matrix $I_q - P^*\Delta Q$ is most extreme if $M = QP^*$ has rank 1.

Lemma IV.4: $\widehat{\mu}_{\mathbb{X}}(M) = \mu_{\mathbb{X}}(M)$ if M has rank 1.

Proof: Let P and Q be column vectors such that $M = QP^*$. By Theorem IV.2 we have that $\widehat{\mu}_{\mathbb{X}}(M) = \widehat{\mu}_{\mathbb{X},2}(P,Q)$. Suppose $\alpha > \mu_{\mathbb{X}}(M)$. Then by definition of $\mu_{\mathbb{X}}$ all members of $1 - P^*(\frac{1}{\alpha}\mathcal{B}_{\mathbb{X}})Q$ are nonzero. Since $1 - P^*(\frac{1}{\alpha}\mathcal{B}_{\mathbb{X}})Q$ is convex it must, therefore, lie in some half-space $\{\, s : \theta - \pi/2 < \arg s < \theta + \pi/2\}$. Then

$$\text{He } e^{-j\theta}(1 - P^*\tfrac{1}{\alpha}\mathcal{B}_{\mathbb{X}}Q) = \text{Re } e^{-j\theta}(1 - P^*\tfrac{1}{\alpha}\mathcal{B}_{\mathbb{X}}Q) > 0,$$

so that $\widehat{\mu}_{\mathbb{X},2}(P,Q) < \alpha$. Since this holds for any $\alpha > \mu_{\mathbb{X}}(M)$ the result follows. ∎

Young [19] showed that also (D,G)-scaling is exact for rank 1 matrices M. The multiplier method is also exact if $\mathbb{X} = \mathbb{C}^{n\times n}$ in which case we can simply take the multiplier to be the identity.

V. Continuity

In typical robust stability applications the nonsingularity property that needs to be tested is

$$I - \tfrac{1}{\alpha}\mathcal{B}_{\mathbb{X}}H(j\omega) \text{ is nonsingular for all } \omega \in \mathbb{R} \cup \infty \text{ ?} \qquad (14)$$

In other words, checking at each frequency ω whether or not $\mu_{\mathbb{X}}(H(j\omega)) < \alpha$. This is usually impossible and one has to rely on a finite frequency grid. The problem however is that $\mu_{\mathbb{X}}$ and also $\widehat{\mu}_{\mathbb{X}}$ and $\nu_{\mathbb{X}}$ are discontinuous which greatly impairs the use of gridding.

Example V.1: Suppose $\mathbb{X} = \mathbb{R}$. If x is a real then $\mu_{\mathbb{X}}(x) = |x|$. $\mu_{\mathbb{X}}$ is discontinuous because for every (arbitrarily small) $y \neq 0$ we have that $\mu_{\mathbb{X}}(x + jy) = 0$. Since $\mu_{\mathbb{X}} = \widehat{\mu}_{\mathbb{X}} = \nu_{\mathbb{X}}$ for this case (See Lemma IV.4) also $\widehat{\mu}_{\mathbb{X}}$ and $\nu_{\mathbb{X}}$ are discontinuous.

Generally $\mu_{\mathbb{X}}(H(j\omega))$ and its upper bounds are discontinuous at some isolated frequencies only. An even more severe discontinuity problem that can occur is that the infimal α for which (14) holds true may be discontinuous as a function of H (measured in, say, the $\mathcal{H}_\infty$ norm). This problem was first demonstrated in [10] and subsequently rigorously studied in [3]. The multiplier method upper bound can be adjusted—at the expense of worsening the bound—to one that is continuous:

Lemma V.2: Let η be some (small) positive number < 1. If the multiplier C in (4) is restricted to the convex set

$$\|C\| < 1 \ \text{ and } \ \text{He } C > \eta I, \qquad (15)$$

then C has condition number less than $1/\eta$ and $\widehat{\mu}_{\mathbb{X}}$ is continuous:

$$|\widehat{\mu}_{\mathbb{X}}(M + E) - \widehat{\mu}_{\mathbb{X}}(M)| \leq \frac{1}{\eta}\|E\|. \qquad (16)$$

The proof is trivial. Note that the condition $\|C\| < 1$ alone has no effect on the upper bound $\widehat{\mu}_{\mathbb{X}}(M)$ since the multiplier may be scaled to be like that. Secondly note that any feasible multiplier C is nonsingular and satisfies He $C > 0$. However as the bound α approaches its optimal value $\widehat{\mu}_{\mathbb{X}}(M)$ the associated feasible multiplier C may become ill-conditioned in the sense that He C approaches a singular matrix. An advantage of the continuity condition (16) over that of Lee and Tits [13] is that (16) is a Lipschitz continuity condition with known Lipschitz constant $1/\eta$.

Example V.3: Suppose the multiplier C is restricted to (15) for some $0 < \eta < 1$. For $\mathbb{X} = \mathbb{R}$ it may be verified that $\widehat{\mu}_{\mathbb{X}}(x + jy) = \max\{0, |x| - |y|\frac{\sqrt{1-\eta^2}}{\eta}\}$, (cf. Example V.1.)

The LMI (15) is easily incorporated in the convex upper bounds of Fan, Tits and Doyle [9] and Fu and Barabanov [11] as all of them generate a feasible multiplier C. (The condition (15) is also an LMI in (D,G) if we use $C = \alpha^2 D + jM^*G$.)

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