The standard $\mathcal{H}_\infty$ control problem for dead-time systems

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Abstract

The standard $\mathcal{H}_\infty$ control problem is solved for systems with a delay. It is shown that for a given bound on the $\mathcal{H}_\infty$-norm there exist causal stabilizing controllers that achieve this bound if and only if two *rational* spectral factorization problems have solutions with a certain nonsingularity property. All sub-optimal controllers are parameterized and the central controller is shown to consist of an interconnection of a finite dimensional system with a system whose impulse response has compact support.

Keywords

Dead-time systems, delay systems, $\mathcal{H}_\infty$ control, infinite dimensional systems, spectral factorization, Riccati equations.

1 Introduction

Dead-time systems are systems where the effects of the control inputs take a certain time before they can be detected in the measured outputs. The typical dead-time system is

$$e^{-s\tau}P_{\mathrm{r}}(s) \tag{1}$$

where P_{r} is some rational matrix. Systems like these are important because of their ability to model, say, chemical processes where often dead-time is present, and because they often serve as simple yet adequate models for otherwise complicated infinite dimensional systems.

In [12] the mixed sensitivity $\mathcal{H}_\infty$ control problem is solved for dead-time systems (1), and in this paper we extend these results to the standard $\mathcal{H}_\infty$ control problem. Other papers in this line of research are Ball and Helton [1], Foias, Özbay and Tannenbaum [4], Toker and Özbay [16], Dym, Georgiou and Smith [3], Partington and Glover [13]. These follow an operator theoretic approach. Time domain approaches may be found in van Keulen [17], Tadmor [15] and Kojima and Ishjima [7]. The results that we present are much influenced by the J-spectral factorization approach for finite dimensional systems [5, 8, 10], see also [2].

For zero time delay, $\tau = 0$, the problem that we consider reduces to the standard $\mathcal{H}_\infty$ control problem for finite dimensional systems. The intersting case is when $\tau > 0$, which renders the problem infinite dimensional. A relatively easy but interesting deviation from the finite dimensional case is that now we have to worry about the causality condition. Indeed a "blind" translation of the spectral factorization results known for the finite dimensional case might lead to controllers that are non-causal.

The main asset of the results that we present is that the controllers, although infinite dimensional, may be constructed from two *finite* dimensional spectral factorization problems, or equivalently, from two finite dimensional Riccati equations. Moreover the controllers turn out consist of a finite dimensional system in feedback with a system whose impulse response has compact support. This is rather fortunate and means that such controllers can be used for simulation and can in fact be build much the same way as Smith predictors are build.

2 Preliminaries

For constant matrices M the spectral norm is denoted by $\|M\|$. The spaces $\mathcal{H}_2$ and $\mathcal{H}_\infty$ denote the standard Hardy spaces defined on the open right-half plane. Their respective norms are interrelated through

$$\|G\|_{\mathcal{H}_\infty} = \sup_{u \in \mathcal{H}_2} \frac{\|Gu\|_{\mathcal{H}_2}}{\|u\|_{\mathcal{H}_2}}.$$

We normally do not mention dimensions and simply write $\mathcal{H}_\infty$ when we mean $\mathcal{H}_\infty^{n \times m}$. For $G \in \mathcal{H}_\infty$ the adjoint $G^\sim$ satisfies $G^\sim(s) = (G(-\bar{s}))^*$. The elements of $\mathcal{H}_\infty$ are called *stable*, and G is said to be *bistable* if $G, G^{-1} \in \mathcal{H}_\infty$. The quotient field of $\mathcal{H}_\infty$ is denoted by $\mathcal{F}_\infty$, that is,

$$\mathcal{F}_\infty^{n \times m} := \{\, H^{-1}G \;:\; G \in \mathcal{H}_\infty^{n \times m},\ H \in \mathcal{H}_\infty^{n \times n},\ \det H \neq 0 \,\}.$$

This field $\mathcal{F}_\infty$ will be our class of transfer matrices. An element $C \in \mathcal{F}_\infty$ is said to have a *strong coprime factorization* if there are stable C_n, C_d, X, Y such that $C = C_\mathrm{d}^{-1} C_\mathrm{n}$ and $C_\mathrm{d} X + C_\mathrm{n} Y = I$. Not all elements of $\mathcal{F}_\infty$ have a strong coprime factorization [14].

Weiss [18] showed that a transfer matrix $C \in \mathcal{F}_\infty$ is causal if there is a ρ such that $C(s-\rho)$ is in $\mathcal{H}_\infty$. The dimension of a signal u is denoted by n_u, i.e., $u(s) \in \mathbb{C}^{n_u}$.

3 A two-block type $\mathcal{H}_\infty$ problem for systems with dead-time

A basic building block in frequency domain $\mathcal{H}_\infty$ theory is the following result. It was used by Green [5] for the finite dimensional case and was given a signal interpretation (also for the finite dimensional case only) in [11]. Here we present the version for dead-time systems as reported in [12].

Definition 3.1. Fix $\tau > 0$. Let $L \in \mathcal{H}_\infty^{(n_z+n_w)\times(n_u+n_w)}$ and consider the equality

$$\begin{bmatrix} H_1(s) \\ H_2(s) \end{bmatrix} = L(s) \begin{bmatrix} e^{-s\tau} C_1(s) \\ C_2(s) \end{bmatrix}$$

with $H_1 \in \mathcal{H}_\infty^{n_z \times n_w},\ H_2 \in \mathcal{H}_\infty^{n_w \times n_w},\ C_1 \in \mathcal{H}_\infty^{n_u \times n_w},\ C_2 \in \mathcal{H}_\infty^{n_w \times n_w}$.

Let $\gamma > 0$. The *two-block problem* is to find a causal controller $C \in \mathcal{F}_\infty$ that has a factorization $C = C_1 C_2^{-1}$ over $\mathcal{H}_\infty$ such that H_2 is bistable and $\|H_1 H_2^{-1}\|_{\mathcal{H}_\infty} < \gamma$. □

Theorem 3.2. *Let $L \in \mathcal{H}_\infty^{(n_z+n_w)\times(n_u+n_w)}$ be a rational transfer matrix that has full column rank on the imaginary axis and at infinity. Fix a time-delay $\tau > 0$.*

The two-block problem has a (causal) solution C if and only if the following three conditions hold.

1. *Π defined as $\Pi := \left[\begin{smallmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{smallmatrix}\right] := L^\sim \left[\begin{smallmatrix} I_{n_z} & 0 \\ 0 & -\gamma^2 I_{n_w} \end{smallmatrix}\right] L$ has full column rank on the imaginary axis and at infinity,*
2. *$\Pi_{22}(\infty) < 0$*
3. *Decompose $e^{-s\tau}\Pi_{22}^{-1}(s)\Pi_{21}(s)$ as $e^{-s\tau}\Pi_{22}^{-1}(s)\Pi_{21}(s) = F_{\mathrm{stab}}(s) + R(s)$, with $F_{\mathrm{stab}} \in \mathcal{H}_\infty$ and R rational. The rational matrix*

$$\Theta := \begin{bmatrix} \Pi_{11} - \Pi_{12}\Pi_{22}^{-1}\Pi_{21} + R^\sim \Pi_{22} R & R^\sim \Pi_{22} \\ \Pi_{22} R & \Pi_{22} \end{bmatrix}$$

admits a spectral factorization $\Theta = Q^\sim \left[\begin{smallmatrix} I_{n_u} & 0 \\ 0 & -I_{n_y} \end{smallmatrix}\right] Q$ with Q bistable, and the lower-right $n_w \times n_w$ block M_{22} of $M(s) := L(s) \left[\begin{smallmatrix} e^{-s\tau} I & 0 \\ -F_{\mathrm{stab}}(s) & I \end{smallmatrix}\right] Q^{-1}(s)$ is bistable.

In that case C solves the two-block problem if and only if

$$C = (Z_{11}U + Z_{12})(Z_{21}U + Z_{22})^{-1}, \tag{2}$$

where $Z := \left[\begin{smallmatrix} I & 0 \\ -F_{\text{stab}} & I \end{smallmatrix}\right] Q^{-1} \in \mathcal{H}_\infty^{(n_u+n_w)\times(n_u+n_w)}$ *and U is stable with* $\|U\|_{\mathcal{H}_\infty} < 1$*. Moreover, Q (which is not unique) can be chosen such that its upper-right* $n_u \times n_w$ *block* Q_{12} *satisfies* $\lim_{s\to\infty} Q_{12}(s) = 0$ *and for that choice of Q the controller (2) is causal for any strictly proper U.*

Proof. See [12]. ∎

If the delay τ is zero, then in Item 3 we may take $R = \Pi_{22}^{-1}\Pi_{21}$ and $F_{\text{stab}} = 0$ in which case Θ equals $L^{\sim} J_\gamma L$. Then Items 1 and 3 recover the well known results for the finite dimensional case [5]. Item 2 is typical for dead-time systems and is required for the existence of a causal C. For nonzero time delay the above results still require only a *rational* spectral factorization problem to be solved. We do not have the space to provide the state space realizations of Q and F_{stab}, but they can be found through computation of a finite dimensional Riccati equation, of dimension equal to that of the well known finite dimensional case, and, importantly, F_{stab} is a system whose impulse response can be shown to be of the form

$$\begin{cases} Ce^{(t-\tau)A}B + (\delta(t-\tau) - \delta(t))D & \text{if } t \in [0^-, \tau^+) \\ 0 & \text{elsewhere} \end{cases}. \tag{3}$$

The block diagram of the central controller (the controller (2) for $U = 0$) is depicted in Figure 1. Here K is a finite dimensional system and F_{stab} is as in (3), and because the impulse response of F_{stab} has compact support, such controllers can be used for simulation and can in fact be build.

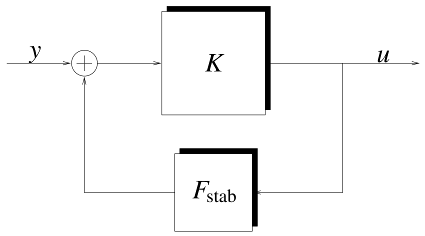

Figure 1: The central controller $C = (I - KF_{\text{stab}})^{-1}K$.

4 The standard $\mathcal{H}_\infty$ problem

In the previous section we reviewed the two-block $\mathcal{H}_\infty$ problem. In this section we see what we can do with it in the standard $\mathcal{H}_\infty$ control problem. The standard $\mathcal{H}_\infty$ control problem configuration for systems with dead-time is shown in Figure 2. The standard $\mathcal{H}_\infty$ control problem for the dead-time system is to find a causal stabilizing $C \in \mathcal{F}_\infty$ such that $\|H\|_{\mathcal{H}_\infty} < \gamma$, where H denotes the closed loop transfer matrix from w to z. We assume that G is rational and proper. Let $G = \begin{bmatrix} D_1 & D_2 \end{bmatrix}^{-1} \begin{bmatrix} N_1 & N_2 \end{bmatrix}$ be a rational coprime factorization of G over $\mathcal{H}_\infty$, where the partitioning of $\begin{bmatrix} D_1 & D_2 \end{bmatrix}$ and $\begin{bmatrix} N_1 & N_2 \end{bmatrix}$ corresponds to the partitionings of the output and input of G. Letting $C = C_{\text{d}}^{-1}C_{\text{n}}$ be a factorization over $\mathcal{H}_\infty$, we can describe the closed loop signal equations of the system of Figure 2 as

$$\begin{bmatrix} -N_1 & D_1 & -N_2 e^{-\cdot\tau} & D_2 \\ 0 & 0 & C_{\text{d}} & -C_{\text{n}} \end{bmatrix} \begin{bmatrix} w \\ z \\ u \\ y \end{bmatrix} = 0.$$

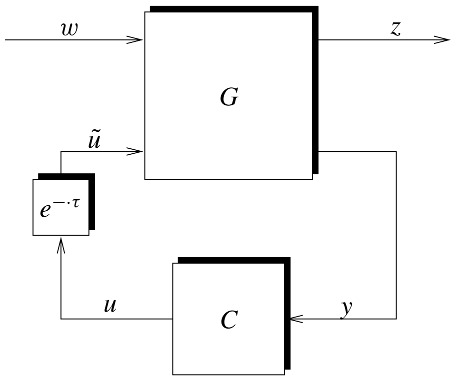

Figure 2: Configuration of the standard $\mathcal{H}_\infty$ control problem with delay.

Under the assumption that G is proper it is not difficult to see that $\begin{bmatrix} D_1 & D_2 \end{bmatrix}^{-1} \begin{bmatrix} N_1 & N_2 e^{-\cdot\tau} \end{bmatrix}$ is a strong coprime factorization[1]. In that case $C \in \mathcal{F}_\infty$ stabilizes iff C has a strong coprime factorization $C = C_{\mathrm{d}}^{-1} C_{\mathrm{n}}$ over $\mathcal{H}_\infty$ and

$$\begin{bmatrix} D_1 & -N_2 e^{-\cdot\tau} & D_2 \\ 0 & C_{\mathrm{d}} & -C_{\mathrm{n}} \end{bmatrix}$$

is bistable.

A key idea in $\mathcal{H}_\infty$ theory is the following idea due to Khargonekar [6] (time domain) and [11, 9] (frequency domain). Consider the set $\mathcal{B}_w$ of signals $w \in \mathcal{L}_2$ for which $y(t)$ is zero for all $t < 0$. For all such signals the feedback signals u and $\tilde{u}$ will be zero for negative times for *every* causal controller C. Hence *necessary* for the existence of a causal, stabilizing γ-optimal controller C is that

$$\sup_{w \in \mathcal{B}_w} \frac{\|z\|_{\mathcal{L}_2(-\infty,0]}}{\|w\|_{\mathcal{L}_2(-\infty,0]}} < \gamma. \tag{4}$$

Theorem 4.1. *Assume that $\begin{bmatrix} N_1 & D_1 \end{bmatrix}$ has full row rank on the imaginary axis and at infinity. Then (4) holds if and only if there is a (rational) bistable W such that*

$$N_1 N_1^{\sim} - \gamma^2 D_1 D_1^{\sim} = W \left[\begin{smallmatrix} I_{n_y} & 0 \\ 0 & -I_{n_z} \end{smallmatrix}\right] W^{\sim}$$

and the lower-right $n_z \times n_z$ block M_{22} of $M := W^{-1} \begin{bmatrix} -N_1 & D_1 \end{bmatrix}$ is bistable.

Proof. See [9, page 57–58]. ∎

It should be clear that (4) and Theorem 4.1 do not depend on the delay block, so we may copy the proof in [9] on the finite dimensional case. Also the reduction of the standard $\mathcal{H}_\infty$ control problem to a two-block $\mathcal{H}_\infty$ problem does not rely on the delay block:

Theorem 4.2. *Assume the conditions of theorem 4.1 and further assume that $\begin{bmatrix} N_2 & D_2 \end{bmatrix}$ has full column rank on the imaginary axis and at infinity. Then a controller $C \in \mathcal{F}_\infty$ is causal, stabilizes and achieves $\|H\|_{\mathcal{H}_\infty} < \gamma$ if and only if it solves the two-block problem of Definition 3.1 for*

$$L := \left[\begin{smallmatrix} 0 & I_{n_z} \\ I_{n_y} & 0 \end{smallmatrix}\right] W^{-1} \begin{bmatrix} -N_2 & D_2 \end{bmatrix}.$$

Proof. The proof for the delay-free case is given in [9, Lemma's 4.1.4 and 4.1.5]. The proof in fact does not rely on the delay block, and, hence, is also valid here. ∎

[1] This follows essentially from Smith [14], see also [12, Lemma 3.2].

Thereby the problem is reduced to the two-block problem discussed in the previous section, and therefore the results of that section about controller structure and computation of controllers also apply to the standard $\mathcal{H}_\infty$ control problem.

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