J-SPECTRAL FACTORIZATION AND EQUALIZING VECTORS

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Abstract

This note presents some basic results on J-spectral factorization and its connection with state space realizations. The assumptions needed are mild and the proofs are given in detail.

Keywords: J-spectral factorization, $\mathcal{H}_\infty$ control theory, Riccati equations.

1 Introduction

The frequency domain equivalent of the time domain Riccati equation is J-spectral factorization. And like Riccati equations play an important role in time domain analysis of control problems, finding a J-spectral factor is often a key step in frequency domain methods.

In a few words, the J-spectral factorization problem is to find rational matrices W that are stable, with stable inverse, such that

$$Z(s) = W^{\mathrm{T}}(-s) \begin{pmatrix} I_q & 0 \\ 0 & -I_p \end{pmatrix} W(s) \quad \text{for all complex } s,$$

for some given Z and indices q and p. A special case is the 'ordinary' spectral factorization problem

$$Z(s) = W^{\mathrm{T}}(-s)W(s).$$

This 'ordinary' problem has a long history in control theory that goes back to Wiener filtering in the mid fifties, and it is nowadays assumed common knowledge that this problem has a solution iff Z is biproper, $Z(s) = Z^{\mathrm{T}}(-s)$ and $Z(j\omega) > 0$ on the imaginary axis, including infinity. The ordinary spectral factorization problem can be tackled using easy arguments only [5]. The general case, which turns up in $\mathcal{H}_\infty$ control, is much more involved. In several papers on $\mathcal{H}_\infty$ control, many results concerning J-spectral factorization have been proved along the way [5, 8, 7]. Connections between Nevanlinna-Pick interpolation and J-spectral factors are explored in [1, 3].

The links established between J-spectral factors and their realizations in terms of solutions of Riccati equations, are all based on the canonical factorization theorem developed in 1979 [4]. Recently some papers appeared devoted solely to the J-spectral factorization problem [10, 11]. In [10] conjugation methods are explored avoiding the use of the 'difficult' canonical factorization theorem. In [11] a review on polynomial methods for J-spectral factorization is given.

It is the aim of this note to show that the canonical factorization theorem has an easy proof. The proof hinges on the so called equalizing vectors. Equalizing vectors turn out to be effective in determining the existence or nonexistence of a J-spectral factorization. Proofs are given in detail.

The related notion of J-losslessness is not considered in this note, for that we refer to [8, 7, 1, 6, 10, 9, 13].

1.1 Notation

$\mathcal{C}_-$, $\mathcal{C}_+$, $\mathcal{C}_0$, $\mathcal{C}$	Open left half complex plane, open right half complex plane, imaginary axis, $\mathcal{C} = \mathcal{C}_- \cup \mathcal{C}_0 \cup \mathcal{C}_+$.
$H^\sim$, H^*, $H^{-\sim}$	$H^\sim(s) = H(-\bar{s})^*$, $H^*(s) = (H(s))^*$ and $H^{-\sim} = (H^{-1})^\sim = (H^\sim)^{-1}$.
$\mathcal{L}_2$, $\mathcal{L}_{2-}$, $\mathcal{L}_{2+}$	$\{w \mid \int_{-\infty}^{\infty} w^*(t)w(t)\,\mathrm{d}t < \infty\}$, $\{w \mid \int_{-\infty}^{0} w^*(t)w(t)\,\mathrm{d}t < \infty,\ w(t) = 0 \text{ for } t > 0\}$, $\{w \mid \int_{0}^{\infty} w^*(t)w(t)\,\mathrm{d}t < \infty,\ w(t) = 0 \text{ for } t < 0\}$.
$\mathcal{H}_2, \mathcal{H}_2^\perp$	Set of vector-valued functions f analytic in $\mathcal{C}_+$ ($\mathcal{C}_-$) such that $\sup_{\sigma>0}\int_{-\infty}^{\infty} \|f(\sigma + j\omega)\|^2\,\mathrm{d}\omega < \infty$ ($\sup_{\sigma<0}\int_{-\infty}^{\infty} \|f(\sigma + j\omega)\|^2\,\mathrm{d}\omega < \infty$).
$\mathcal{L}_2(\mathcal{C}_0)$	$\{f \mid \int_{-\infty}^{\infty} f^*(j\omega)f(j\omega)\,\mathrm{d}\omega < \infty\}$.
$\langle f, g\rangle$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} f^*(j\omega)g(j\omega)\,\mathrm{d}\omega$.
$G \overset{\mathrm{s}}{=} \left[\begin{array}{c\|c} A & B \\ \hline C & D \end{array}\right]$, or, $G \overset{\mathrm{s}}{=} [A; B; C; D]$	G has a realization $G(s) = C(sI - A)^{-1}B + D$.
π_+, π_-	Orthogonal projection from $\mathcal{L}_2(\mathcal{C}_0)$ to $\mathcal{H}_2$, $\pi_- = 1 - \pi_+$ (i.e. the orthogonal projection to $\mathcal{H}_2^\perp$).

A rational matrix is *stable* if it has no poles in $\mathcal{C}_+ \cup \mathcal{C}_0 \cup \infty$. G is *bistable* if both G and G^{-1} are stable. We call a square constant matrix A *stable* if its eigenvalues lie in $\mathcal{C}_-$. This presumably will not give rise to confusion. A rational matrix G is *antistable* if $G^\sim$ is stable; a constant square matrix A is called *antistable* if $-A$ is stable.

We say that a realization $G \overset{\mathrm{s}}{=} [A; B; C; D]$ is stable if its "A-matrix" is stable. The *zeros* of a realization $[A; B; C; D]$ are those values s for which the system matrix

$$\begin{pmatrix} A - sI & B \\ C & D \end{pmatrix}$$

drops below normal rank. We define $J_{r,p}$ as

$$J_{r,p} = \begin{pmatrix} I_r & 0 \\ 0 & -I_p \end{pmatrix}.$$

We use J instead of $J_{r,p}$ if the values of r and p are obvious or irrelevant.

1.2 Preliminaries

We make use of the following well known results. Suppose G has realization $G \stackrel{s}{=} [A; B; C; D]$, then

$$G^{\sim} \stackrel{s}{=} \left[\begin{array}{c|c} -A^* & C^* \\ \hline -B^* & D^* \end{array}\right],$$

and, if G is biproper,

$$G^{-1} \stackrel{s}{=} \left[\begin{array}{c|c} A - BD^{-1}C & BD^{-1} \\ \hline -D^{-1}C & D^{-1} \end{array}\right].$$

For any constant matrix J,

$$G^{\sim}JG \stackrel{s}{=} \left[\begin{array}{cc|c} A & 0 & B \\ -C^*JC & -A^* & -C^*JD \\ \hline D^*JC & B^* & D^*JD \end{array}\right].$$

If $G^{\sim}JG$ is biproper, we may, as before, determine a realization of its inverse:

$$(G^{\sim}JG)^{-1} \stackrel{s}{=} \left[\begin{array}{c|c} \begin{pmatrix} A & 0 \\ -C^*JC & -A^* \end{pmatrix} - \begin{pmatrix} B \\ -C^*JD \end{pmatrix}(D^*JD)^{-1}\begin{pmatrix} D^*JC & B^* \end{pmatrix} & * \\ \hline * & * \end{array}\right].$$

That is, its "A-matrix" is a Hamiltonian matrix.

The Hardy spaces $\mathcal{H}_2$ and $\mathcal{H}_2^{\perp}$ are the Laplace transforms of $\mathcal{L}_{2+}$ and $\mathcal{L}_{2-}$, respectively. All steps in this note remain valid if $\mathcal{H}_2$ ($\mathcal{H}_2^{\perp}$) is understood as the set of strictly proper stable (antistable) rational functions. In the same way, $\mathcal{L}_2(\mathcal{C}_0)$ may be understood as the set of strictly proper rational functions without poles on the imaginary axis.

If a rational matrix G is proper and has no poles on the imaginary axis, then $G\,\mathcal{L}_2(\mathcal{C}_0) \subset \mathcal{L}_2(\mathcal{C}_0)$. If G is stable we have $G\,\mathcal{H}_2 \subset \mathcal{H}_2$, and if G is antistable we have $G\,\mathcal{H}_2^{\perp} \subset \mathcal{H}_2^{\perp}$.

2 J-spectral factorization

In this section we provide some if-and-only-if conditions for the existence of a J-spectral factorization. At the end similar results on *co*factorization are given.

Definition 2.1 *A matrix W is a J-spectral factor of a rational matrix Z if W is bistable and*

$$Z = W^{\sim} J W.$$

The factorization $Z = W^{\sim} J W$ is then referred to as a J-spectral factorization. A matrix $\bar{W}$ is a J-spectral cofactor of Z if $\bar{W}$ is bistable and $Z = \bar{W} J \bar{W}^{\sim}$. The factorization $Z = \bar{W} J \bar{W}^{\sim}$ is then a J-spectral cofactorization of Z. ◇

If Z has a $J_{q,p}$-spectral factorization

$$Z = W^{\sim} J_{q,p} W; \qquad W \text{ bistable},$$

then necessarily Z satisfies:

- $Z = Z^{\sim}$;
- Z has no poles and zeros on $\mathcal{C}_0 \cup \infty$;
- Everywhere on the imaginary axis, $Z(j\omega)$ has q positive and p negative eigenvalues.

There is one more necessary condition that we need and which is slightly less obvious. A rational matrix Z has a $J_{q,p}$-spectral factorization only if

- There do not exist nonzero $\hat{u}$ in $\mathcal{H}_2$ for which $Z\hat{u}$ is in $\mathcal{H}_2^{\perp}$.

An equivalent statement is that the Toeplitz operator T_Z with symbol Z has to be injective on its domain $\mathcal{H}_2$, (with $\mathrm{T}_Z(\hat{u}) := \pi_+(Z\hat{u})$). This is easy to check: Suppose Z has a $J_{q,p}$-spectral factorization $Z = W^{\sim} J_{q,p} W$ and that $\hat{y} := Z\hat{u}$ is in $\mathcal{H}_2^{\perp}$ with $\hat{u} \in \mathcal{H}_2$ and nonzero. Then

$$W^{-\sim} \hat{y} = J_{q,p} W \hat{u}.$$

The left-hand side of this equality is in $\mathcal{H}_2^{\perp}$ and the right-hand side is in $\mathcal{H}_2$ and nonzero. This is a contradiction, hence, such $\hat{u}$ do not exist. These vectors $\hat{u}$ appear to be very useful in proving the existence or nonexistence of a J-spectral factorization.

Definition 2.2 *A vector $\hat{u}$ is an equalizing vector of Z if $\hat{u}$ is a nonzero element of $\mathcal{H}_2$ and $Z\hat{u}$ is in $\mathcal{H}_2^{\perp}$.* ◇

Lemma 2.3 (see [7]) *$J_{q,p}$-spectral factors are unique up to multiplication from the left by a constant $J_{q,p}$-unitary[1] matrix.* ◇

Proof. If W and $\bar{W}$ are two $J_{q,p}$-spectral factors of the same matrix $W^{\sim} J_{q,p} W = \bar{W}^{\sim} J_{q,p} \bar{W}$, then $E := W\bar{W}^{-1}$ is bistable and $E^{\sim} J_{q,p} E = J_{q,p}$. Therefore $E^{\sim} J_{q,p} = J_{q,p} E^{-1}$. The left-hand side of $E^{\sim} J_{q,p} = J_{q,p} E^{-1}$ is antistable and the right-hand side is stable, hence, E is constant. The converse is trivial. ■

The main result is presented next. Roughly speaking, it says that the four necessary conditions derived just now, are sufficient as well for a factorization to exist. We assume for the moment that the matrix Z to be factored is given as $Z = G^{\sim} J G$ for some stable (possibly nonsquare) G and some signature matrix J.

[1] A constant matrix E is by definition $J_{q,p}$-*unitary* if $E^* J_{q,p} E = J_{q,p}$.

Theorem 2.4 *Suppose G is a stable rational matrix that has full column rank on $\mathcal{C}_0 \cup \infty$. Let $G(s) \overset{s}{=} [A; B; C; D]$ be a (possibly nonminimal) stable realization without zeros on the imaginary axis. Let n denote the dimension of the state space. The following statements are equivalent.*

1. *$G^{\sim}JG$ has a $J_{q,p}$-spectral factorization for some (unique) $J_{q,p}$.*
2. *$G^{\sim}JG$ has no poles and zeros on $\mathcal{C}_0 \cup \infty$, and has no equalizing vectors.*
3. *D^*JD is nonsingular, and*

$$H := \begin{pmatrix} A & 0 \\ -C^*JC & -A^* \end{pmatrix} - \begin{pmatrix} B \\ -C^*JD \end{pmatrix} (D^*JD)^{-1} \begin{pmatrix} D^*JC & B^* \end{pmatrix} \quad \in \mathcal{C}^{2n\times 2n} \tag{1}$$

   has no imaginary eigenvalues and its stable eigenspace is of the form $\mathrm{Im}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ with $X_1, X_2 \in \mathcal{C}^{n\times n}$ and X_1 nonsingular.

4. *D^*JD is nonsingular, and the Riccati equation*

$$PA + A^*P - [PB + C^*JD](D^*JD)^{-1}[D^*JC + B^*P] + C^*JC = 0 \tag{2}$$

   has a solution P with $A - B(D^*JD)^{-1}[D^*JC + B^*P]$ stable.

In the case that the conditions above are satisfied we have that q is the number of positive eigenvalues of D^*JD, p is the number of negative eigenvalues of D^*JD, and W is a $J_{q,p}$-spectral factor of $G^{\sim}JG$ if and only if

$$W \overset{s}{=} \left[\begin{array}{c|c} A & B \\ \hline J_{q,p}W_\infty^{-*}[D^*JC + B^*P] & W_\infty \end{array}\right], \tag{3}$$

in which W_∞ is a solution of

$$D^*JD = W_\infty^* J_{q,p} W_\infty. \tag{4}$$

◇

Proof. [(1) $\Longrightarrow$ (2)] See the discussion prior to this theorem.

[(2) $\Longrightarrow$ (3)] That D^*JD is nonsingular is obvious. Next we show that H does not have imaginary eigenvalues. Define $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ as

$$G^{\sim}JG \overset{s}{=} \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right] := \left[\begin{array}{cc|c} A & 0 & B \\ -C^*JC & -A^* & -C^*JD \\ \hline D^*JC & B^* & D^*JD \end{array}\right]. \tag{5}$$

Then

$$(G^{\sim}JG)^{-1} \overset{s}{=} \left[\begin{array}{c|c} H & \bar{B}\bar{D}^{-1} \\ \hline -\bar{D}^{-1}\bar{C} & \bar{D}^{-1} \end{array}\right].$$

(Note that the Hamiltonian H defined in (1) equals $H = \bar{A} - \bar{B}\bar{D}^{-1}\bar{C}$.) This realization of $(G^{\sim}JG)^{-1}$ has no unobservable or uncontrollable modes on the imaginary axis because A is stable. By assumption $G^{\sim}JG$ has no zeros on the imaginary axis, and, therefore, H has no imaginary eigenvalues. Since H is a Hamiltonian matrix we get as a result that the stable eigenspace of H is n-dimensional. That is, the stable eigenspace of H is of the form $\mathrm{Im}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ with $X_1, X_2 \in \mathcal{C}^{n\times n}$. The crucial step in the proof is to show that X_1 is nonsingular and this is where we use that $G^{\sim}JG$ has no equalizing vectors.

Proof by contradiction: Suppose X_1 is singular. The antistable eigenspace of $\bar{A}$ is the image of $\begin{pmatrix} 0 \\ I \end{pmatrix}$ because

$$\bar{A} = \begin{pmatrix} A & 0 \\ -C^*JC & -A^* \end{pmatrix},$$

and A is stable. The intersection of the antistable eigenspace of $\bar{A}$ and the stable eigenspace of H

$$\mathrm{Im}\begin{pmatrix} 0 \\ I \end{pmatrix} \cap \mathrm{Im}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

is nontrivial because X_1 is singular. Take a nonzero x_0 from this intersection. It is easy to check that u and y defined as

$$u(t) = \begin{cases} 0 & \text{for } t < 0, \\ -\bar{D}^{-1}\bar{C}x(t) & \text{for } t > 0 \text{ and with } \dot{x} = Hx,\ x(0) = x_0, \end{cases}$$

$$y(t) = \begin{cases} \bar{C}x(t) & \text{for } t < 0 \text{ and with } \dot{x} = \bar{A}x,\ x(0) = x_0, \\ 0 & \text{for } t > 0, \end{cases}$$

satisfy the differential equation

$$\begin{pmatrix} \dot{x} \\ y \end{pmatrix} = \begin{pmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{pmatrix}\begin{pmatrix} x \\ u \end{pmatrix}, \quad x(0) = x_0, \tag{6}$$

or, equivalently, the differential equation

$$\begin{pmatrix} \dot{x} \\ u \end{pmatrix} = \begin{pmatrix} H & \bar{B}\bar{D}^{-1} \\ -\bar{D}^{-1}\bar{C} & \bar{D}^{-1} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \quad x(0) = x_0. \tag{7}$$

Both u and y are in $\mathcal{L}_2$ because x_0 is as well in the stable eigenspace of H as in the antistable eigenspace of $\bar{A}$. The Laplace transforms $\hat{u}$ and $\hat{y}$ of u and y are, therefore, well defined on the imaginary axis; they are strictly proper rational funtions and satisfy $\hat{y} = G^{\sim}JG\hat{u}$. Since $u(t)$ is zero for negative time and $y(t)$ is zero for positive time we have that $\hat{u}$ is in $\mathcal{H}_2$ and $\hat{y}$ in $\mathcal{H}_2^{\perp}$, i.e, that $\hat{u}$ is an equalizing vector. By Item 2 such vectors do not exist. This is a contradiction and, hence, X_1 is nonsingular. We glossed over the possibility that $\hat{u}$ is the zero function. This can not be: For $t > 0$ we have $\dot{x} = Hx = (\bar{A} - \bar{B}\bar{D}^{-1}\bar{C})x = \bar{A}x + \bar{B}u$. Therefore u is identically zero only if $\dot{x} = \bar{A}x = Hx$ for $t > 0$. This is clearly impossible as $x(0) = x_0 \neq 0$ is in the antistable eigenspace of $\bar{A}$ and in the stable eigenspace of H.

[(3) $\Longrightarrow$ (4)] Standard result: Define $P := X_2 X_1^{-1}$, then $\binom{I}{P}$ spans the stable eigenspace of H, so that

$$H \begin{pmatrix} I \\ P \end{pmatrix} = \begin{pmatrix} I \\ P \end{pmatrix} \Lambda \tag{8}$$

with Λ stable. Obviously $\Lambda = A - B(D^*JD)^{-1}[D^*JC + B^*P]$. Multiplying (8) from the left by $(-P \quad I)$ shows $(-P \quad I) H \binom{I}{P} = 0$. This, expressed term by term, is the Riccati equation.

[(3) $\Longrightarrow$ (1)] Standard result (see [7]) and easy to check.

Finally we show that W is a $J_{q,p}$-spectral factor *only if* it has a realization as in (3-4). The solutions W_∞ of (4) are unique up to multiplication from the left by a (constant) $J_{q,p}$-unitary matrix. The same holds for W defined in (3), which is easier to see if we rewrite realization (3) as

$$W \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline W_\infty (D^*JD)^{-1}[D^*JC + B^*P] & W_\infty \end{array} \right].$$

By Lemma 2.3 this comprises *all* $J_{q,p}$-spectral factors. ■

Without proof we give the conjugate version:

Theorem 2.5 *Suppose G is a stable rational matrix that has full row rank on $\mathcal{C}_0 \cup \infty$. Let $G(s) \stackrel{s}{=} [A; B; C; D]$ be a (possibly nonminimal) stable realization without zeros on the imaginary axis. Then $GJG^\sim$ has a $J_{q,p}$-spectral cofactorization if and only if DJD^* has q positive and p negative eigenvalues and there exists a (unique) $\bar{Q}$ such that*

$$A\bar{Q} + \bar{Q}A^* - [\bar{Q}C^* + BJD^*](DJD^*)^{-1}[C\bar{Q} + DJB^*] + BJB^* = 0 \tag{9}$$

with $A - [\bar{Q}C^* + BJD^*](DJD^*)^{-1}C$ stable. In this case $\bar{W}$ is a $J_{q,p}$-spectral cofactor if and only if

$$\bar{W} \stackrel{s}{=} \left[\begin{array}{c|c} A & [BJD^* + QC^*]\bar{W}_\infty^{-*} J_{q,p} \\ \hline C & \bar{W}_\infty \end{array} \right]$$

in which $\bar{W}_\infty$ is a solution of $\bar{W}_\infty J_{q,p} \bar{W}_\infty^* = DJD^*$. ◇

3 Three corollaries

Corollary 3.1 *A rational matrix Z has a $J_{q,p}$-spectral factorization for some (unique) $J_{q,p}$ if and only if the following three conditions hold.*

1. *$Z = Z^\sim$.*

2. *Z has no poles and zeros on $\mathcal{C}_0 \cup \infty$.*

3. *Z has no equalizing vectors.*

◇

Proof. Necessity is shown in the previous section.

(Sufficiency) Any $Z = Z^\sim$ without poles and zeros on $\mathcal{C}_0 \cup \infty$ may be written as $Z = G^\sim JG$ for some J and stable G without zeros on $\mathcal{C}_0 \cup \infty$. For example, if T is the stable part of Z and $Z = T + T^\sim$, then $Z = G^\sim J_{m,m} G$ with $G := \frac{1}{2}\begin{pmatrix} I+T \\ I-T \end{pmatrix}$. The result then follows from Theorem 2.4, Items 1 and 2. ■

Corollary 3.2 (Ordinary spectral factorization) *A $Z = Z^\sim$ without poles and zeros on $\mathcal{C}_0 \cup \infty$ has a spectral factorization*

$$Z = W^\sim W$$

if and only if $Z(i\omega) > 0$ for all $\omega \in \mathcal{R}$. ◇

Proof. (If) We only need to show that Z has no equalizing vectors. If $\hat{u}$ is an equalizing vector of Z, then $\langle Z\hat{u}, \hat{u}\rangle = 0$ because $\hat{u} \in \mathcal{H}_2$ and $Z\hat{u} \in \mathcal{H}_2^\perp$ are perpendicular. On the other hand $\langle Z\hat{u}, \hat{u}\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}^*(j\omega)Z(j\omega)\hat{u}(j\omega)\,\mathrm{d}\omega$ is positive because $Z(j\omega) > 0$ by assumption. This is a contradiction, hence, Z has no equalizing vectors and, therefore, has a $J_{q,p}$-spectral factorization $Z = W^\sim J_{q,p} W$ for some $J_{q,p}$. In this case $p = 0$ because $Z(j\omega) > 0$. In other words, $J_{q,p} = I$. The converse is trivial. ■

Corollary 3.3 *Let G be a stable rational matrix that has full column rank on $\mathcal{C}_0 \cup \infty$ and suppose the realization $G \stackrel{\mathrm{s}}{=} [A; B; C; D]$ is stable, controllable and without zeros on the imaginary axis. Then*

$$G^\sim JG = J_{q,p}$$

if and only if there exist (unique) P such that

1. $D^*JD = J_{q,p}$;
2. $D^*JC + B^*P = 0$;
3. $PA + A^*P + C^*JC = 0$.

◇

Proof. We use the fact that $G^\sim JG = J_{q,p}$ iff $W = I$ is a $J_{q,p}$-spectral factor of $G^\sim JG$.

(If) Items 1,2 and 3 imply that P is also the stabilizing solution of (2). (Note that by Item 2 we have $A - B(D^*JD)^{-1}[D^*JC + B^*P] = A$, so the stability condition in Theorem 2.4, Item 4 is void.) Therefore W in (3) is constant: $W = W_\infty$ and because $D^*JD = J_{q,p}$ we may take $W = W_\infty = I$.

(Only if) That $D^*JD = J_{q,p}$ is obvious. Let P be the solution of (2). By controllability of (A, B), the $J_{q,p}$-spectral factor W in (3) is constant iff $[D^*JC + B^*P] = 0$. The Riccati equation (2) then reduces to the Lyapunov equation in Item 3. ■

References

[1] J. A. Ball, I. Gohberg and L. Rodman, *Interpolation of Rational Matrix Functions,* Operator Theory: Advances and Appl., **45** (Birkhäuser Verlag, Basel, 1990).

[2] J. A. Ball and N. Cohen, Sensitivity minimization in an $\mathcal{H}^\infty$ norm, *Int. J. Control* **46** (1987) 785-816.

[3] J. A. Ball and J. W. Helton, Shift invariant subspaces, passivity, reproducing kernels and $\mathcal{H}^\infty$-optimization, in: *Operator Theory: Adv. Appl.,* **35** (Birkhäuser Verlag, Basel, 1988) 265-310.

[4] H. Bart, I. Gohberg and M. A. Kaashoek, *Minimal Factorization of Matrix and Operator Functions* Operator Theory: Advances and Appl., **1**, (Birkhäuser Verlag, Basel, 1979).

[5] B. A. Francis, *A Course in* $\mathcal{H}_\infty$ *Control Theory,* Springer Lecture Notes in Control and Information Sciences **88** (Springer-Verlag, Heidelberg, 1987).

[6] Y. Genin, P. van Dooren and T. Kailath, On Σ-lossless transfer functions and related questions, *Linear Algebra and its Applications* **50** (1983) 251-275.

[7] M. Green, K. Glover, D. J. N. Limebeer and J. C. Doyle, A J-spectral factorization approach to $\mathcal{H}_\infty$ control, *SIAM J. Control and Opt.* **28** (1990) 1350-1371.

[8] M. Green, $\mathcal{H}_\infty$ controller synthesis by J-lossless coprime factorization, *SIAM J. Control and Opt.* **30** (1992) 522-547.

[9] H. Kimura, Y. Lu and R. Kawatani, On the structure of $\mathcal{H}^\infty$ control systems and related extensions, *IEEE Trans. Aut. Control* **36** (1991) 653-667.

[10] H. Kimura, (J, J')-lossless factorization based on conjugation, *Systems Control. Lett.* **19** (1992) 95-109.

[11] H. Kwakernaak and M. Šebek, Polynomial J-spectral factorization, *IEEE Trans. Aut. Control* **39** (1994) 315-328.

[12] G. Meinsma, *Frequency Domain Methods in* $\mathcal{H}_\infty$ *Control,* Ph.D. Dissertation, Dept. Appl. Mathematics, Univ. of Twente, The Netherlands, 1993.

[13] G. Meinsma, J-spectral factorization and equalizing vectors—the extended remix, Technical Report EE9404, Dept. of Electrical and Comp. Engineering, The University of Newcastle, 1994.