# ON THE STANDARD $H_2$ PROBLEM

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Abstract: This note presents frequency domain solutions of the standard MIMO $H_2$ problem and some non-standard $H_2$ problems. In the standard case the results are equivalent to the well known state space formulae. The non-standard case is a fairly straightforward generalization of the standard case and relies on factorizations over polynomial matrices and stable matrices. Also, a frequency domain formulation of LQR is explored.

Keywords: LQG control, LQR control method, $H_2$ control, frequency domains, polynomial methods

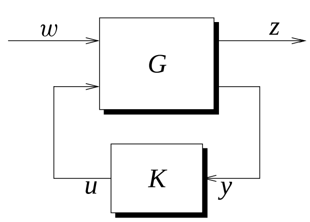


Fig. 1. The standard system configuration.

## 1. INTRODUCTION

The LQG problems and related $H_2$ problems have transparent solutions in the time domain (see e.g. (Doyle *et al.*, 1989), (Zhou *et al.*, 1996)), but it seems that frequency domain solutions are more involved (e.g. (Park and Bongiorno, 1989; Hunt *et al.*, 1994; Kučera, 1996)). The situation with $H_\infty$ is quite different. The $H_\infty$ control problems have a very appealing frequency domain solution, which can stand the test with time domain solutions, and in fact beat time domain solutions in its compactness and generality, (Green, 1992; Kwakernaak, 1993).

As $H_2$ and $H_\infty$ are intimately related, it is tempting to think that it is possible to develop a frequency domain solution of the standard $H_2$ problem along the same lines as that of the $H_\infty$ solution. That we do in this note. The solution of the $H_2$ problem that we present is straightforward and quite general. Computationally it is not essentially easier than existing frequency domain solutions; the method still requires two spectral factorizations and one or two projections. The state space solution applies to a restricted set of problems, but when it does apply it has a definite advantage in that the method does not require computation of projections. It is not clear whether this comprises a fundamental difference between frequency domain and time domain, or a shortcoming of the present frequency domain approaches.

### 1.1 *Notation*

The frequency domain $L_2$-norm and $H_2$-norm are defined as

$$\|H\|_2 := \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} H^{\sim}(\mathrm{j}\omega)H(\mathrm{j}\omega)\,d\omega},$$

where $H^{\sim}(s) := H(-s)^*$. Elements of $H_\infty$ are called *stable*. A stable transfer matrix is said to be *bistable* if also its inverse is stable. For rational functions $H$ without imaginary poles we denote by $\{H\}_+$ and $\{H\}_-$ its strictly proper stable part and strictly proper antistable part respectively. The polynomial part is denoted by $\{H\}_\infty$, that is, $\{H\}_\infty = H - \{H\}_+ - \{H\}_-$.

We say that a rational matrix $M$ is *polynomial-stable* if its strictly proper antistable part $\{M\}_-$ is zero. A polynomial-stable matrix is hence a sum of a polynomial matrix and a stable rational matrix. If both $M$ and $M^{-1}$ are polynomial-stable, then we say that $M$ is *bi-polynomial-stable*.

## 2. A FUNDAMENTAL LEMMA

A fundamental result in frequency domain approaches to $H_\infty$ is the so called two-block problem. Also in the $H_2$ case a fundamental role is played by a two-block problem, but now of course in the $H_2$-norm. It is a standard projection result:

*Lemma 2.1.* Suppose that $A$ and $B$ are polynomial-stable matrices with equally many rows. Suppose that

(1) $B^{\sim}B = I$,
(2) $B^{\sim}A$ is strictly proper and has no stable poles.

Then for any polynomial-stable $Q$ we have that

$$\|A+BQ\|_2^2 = \|A\|_2^2 + \|Q\|_2^2.$$

Therefore $\|A+BQ\|_2$ is finite for some stable $Q$ iff $\|A\|_2$ is finite and then the unique polynomial-stable $Q$ that minimizes $\|A+BQ\|_2$ is $Q=0$.

**PROOF.** By assumption $B^{\sim}B = I$. Then there exists a so-called inner completion $T$ such that $\begin{bmatrix} B & T\end{bmatrix}$ is square and $\begin{bmatrix} B^{\sim} \\ T^{\sim}\end{bmatrix}\begin{bmatrix} B & T\end{bmatrix} = I$. There holds that

$$\|A+BQ\|_2^2 = \|\begin{bmatrix} B^{\sim} \\ T^{\sim}\end{bmatrix}(A+BQ)\|_2^2 = \left\|\begin{bmatrix} B^{\sim}A+Q \\ T^{\sim}A\end{bmatrix}\right\|_2^2.$$

By assumption, $B^{\sim}A$ is strictly proper so the above 2-norm is finite only if $Q$ is strictly proper. Further since $\{B^{\sim}A\}_+ = 0$ and $\{Q\}_- = 0$ it follows that $\|A+BQ\|_2^2 = \|\begin{bmatrix} B^{\sim} \\ T^{\sim}\end{bmatrix}A\|_2^2 + \|Q\|_2^2 = \|A\|_2^2 + \|Q\|_2^2$. ■

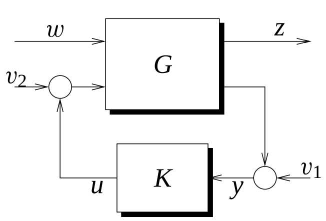


Fig. 2. The standard system configuration with disturbances.

## 3. THE STANDARD $H_2$ PROBLEM

Consider the loop in Fig. 1. The standard $H_2$ problem is to minimize $\|H\|_2$ over the stabilizing controllers $K$. Here $H$ is the transfer matrix from $w$ to $z$ and stability is in this section taken to be that all closed loop poles are in the open left-half plane and all maps from $w$, $v_1$, $v_2$ to $z$, $u$, $y$ are proper (see Fig. 2).

We assume that the transfer function $G$ and $K$ are rational. Now take any coprime factorization over the stable matrices, of plant and controller

$$G = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22}\end{bmatrix}^{-1}\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22}\end{bmatrix}, \quad K = X^{-1}Y. \quad (1)$$

We shall assume the "standard assumptions" which in terms of the coprime factorization are [1]:

**(A1)** $\begin{bmatrix} -N_{11} & D_{11} \\ -N_{21} & D_{21}\end{bmatrix}$ has full row rank on $j\mathbb{R}\cup\infty$,
**(A2)** $\begin{bmatrix} D_{21} & -N_{12} \\ D_{22} & -N_{22}\end{bmatrix}$ has full column rank on $j\mathbb{R}\cup\infty$,
**(A3)** The closed loop is stabilizable by some $K$.

The stabilizability assumption implies that we may shape the coprime factorization a bit. To see this we first describe the closed loop of Fig. 2:

$$\underbrace{\begin{bmatrix} D_{11} & D_{12} & -N_{12} \\ D_{21} & D_{22} & -N_{22} \\ 0 & -Y & X\end{bmatrix}}_{\Omega:=}\begin{bmatrix} z \\ y \\ u\end{bmatrix} = \underbrace{\begin{bmatrix} N_{11} & D_{12} & N_{12} \\ N_{21} & D_{22} & N_{22} \\ 0 & 0 & 0\end{bmatrix}}_{\Psi:=}\begin{bmatrix} w \\ v_1 \\ v_2\end{bmatrix}.$$

Stability is equivalent to bistability of the rational matrix $\Omega$. In particular this shows that $\begin{bmatrix} D_{11} \\ D_{21}\end{bmatrix}$ has no unstable zeros, so we might have chosen our coprime factorization right from the start in such a way that $\begin{bmatrix} D_{11} \\ D_{21}\end{bmatrix} = \begin{bmatrix} I \\ 0\end{bmatrix}$. This gives for the closed loop that

$$\begin{bmatrix} I & D_{12} & -N_{12} \\ 0 & D_{22} & -N_{22} \\ 0 & -Y & X\end{bmatrix}\begin{bmatrix} z \\ y \\ u\end{bmatrix} = \begin{bmatrix} N_{11} & D_{12} & N_{12} \\ N_{21} & D_{22} & N_{22} \\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix} w \\ v_1 \\ v_2\end{bmatrix} \quad (2)$$

In this form stabilizability is equivalent to the existence of a bistable $U$ such that $\begin{bmatrix} D_{22} & -N_{22}\end{bmatrix}U = \begin{bmatrix} I & 0\end{bmatrix}$.

*Theorem 3.1.* Suppose $G$ satisfies assumptions A1, A2, A3. Then there is a coprime factorization of $G$ such that $\begin{bmatrix} D_{11} \\ D_{21}\end{bmatrix} = \begin{bmatrix} I \\ 0\end{bmatrix}$ and

(1) $N_{21}N_{21}^{\sim} = I$,
(2) $N_{11}N_{21}^{\sim}$ has no stable poles and is strictly proper.

Also there is then a bistable $U$ such that $A$, $B$ defined as

$$\begin{bmatrix} A & B \\ I & 0\end{bmatrix} := \begin{bmatrix} D_{12} & -N_{12} \\ D_{22} & -N_{22}\end{bmatrix}U$$

satisfy

(3) $B^{\sim}B = I$,
(4) $B^{\sim}A$ has no stable poles and is strictly proper.

With these data, $K$ is stabilizing iff $K = (U_{21}+U_{22}Q)(U_{11}+U_{12}Q)^{-1}$ for some stable $Q$, and then

$$\|H\|_2^2 = \|N_{11}\|_2^2 + \|A\|_2^2 + \|Q\|_2^2.$$

In particular $\|H\|_2$ can be made finite if and only if both $N_{11}$ and $A$ are strictly proper, in which case the

[1] If $G(s)$ is proper and has minimal realization $G(s) = \begin{bmatrix} C_1 \\ C_2\end{bmatrix}(sI-A)^{-1}\begin{bmatrix} B_1 & B_2\end{bmatrix} + \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22}\end{bmatrix}$ then Assumption A1 is equivalent to $\begin{bmatrix} A-sI & B_1 \\ C_2 & E_{21}\end{bmatrix}$ having full row rank on $j\mathbb{R}\cup\infty$, and Assumption A2 is equivalent to $\begin{bmatrix} A-sI & B_2 \\ C_1 & E_{12}\end{bmatrix}$ having full column rank on $j\mathbb{R}\cup\infty$.

stabilizing controller $K$ that minimizes $\|H\|_2$ is given by $Q = 0$ (i.e., $K = U_{21}U_{11}^{-1}$).

**PROOF.** The proof is practically an algorithm.

(a) By Assumption A1 we have that $N_{21}$ has full row rank on the imaginary axis so there is a bistable $V$ such that $VV^{\sim} = N_{21}N_{21}^{\sim}$. With it we redefine the coprime factorization:

$$\begin{bmatrix} I & D_{12} & N_{11} & N_{12} \\ 0 & D_{22} & N_{21} & N_{22} \end{bmatrix} := \begin{bmatrix} I & 0 \\ 0 & V^{-1} \end{bmatrix} \begin{bmatrix} I & D_{12} & N_{11} & N_{12} \\ 0 & D_{22} & N_{21} & N_{22} \end{bmatrix}$$

(Now $N_{21}N_{21}^{\sim} = I$.)

(b) Let [2] $F := -\{N_{11}N_{21}^{\sim}\}_{+,\infty}$. Then $(N_{11} + FN_{21})N_{21}^{\sim}$ has no stable poles and is strictly proper. So $N_{11} := N_{11} + FN_{21}$ satisfies Item 2. Therefore redefine the coprime factorization,

$$\begin{bmatrix} I & D_{12} & N_{11} & N_{12} \\ 0 & D_{22} & N_{21} & N_{22} \end{bmatrix} := \begin{bmatrix} I & F \\ 0 & I \end{bmatrix} \begin{bmatrix} I & D_{12} & N_{11} & N_{12} \\ 0 & D_{22} & N_{21} & N_{22} \end{bmatrix}.$$

Now we have a coprime factorization that meets the conditions of Items 1 and 2. The remaining conditions (items 3 and 4) follow from a dual version, shown next.

From (2) we can see that a controller is stabilizing iff $K$ has a right coprime factorization $K = \bar{Y}\bar{X}^{-1}$ such that $D_{22}\bar{X} - N_{22}\bar{Y} = I$. Given such a $(\bar{Y}, \bar{X})$ define $Q_K$ via

$$\begin{bmatrix} Q_K \\ I \end{bmatrix} := \begin{bmatrix} D_{12} & -N_{12} \\ D_{22} & -N_{22} \end{bmatrix} \begin{bmatrix} \bar{X} \\ \bar{Y} \end{bmatrix}. \qquad (3)$$

It is easy to verify from (2) that then the closed loop transfer matrix equals $H = N_{11} - Q_K N_{21}$. By Lemma 2.1 we thus have that

$$\|H\|_2^2 = \|N_{11}\|_2^2 + \|Q_K\|_2^2. \qquad (4)$$

Hence minimizing $\|H\|_2$ amounts to minimizing $\|Q_K\|_2$. We shall write (3) in a more manageable form:

(c) By stabilizability there is a bistable $W$ such that $\begin{bmatrix} D_{22} & -N_{22} \end{bmatrix} W = \begin{bmatrix} I & 0 \end{bmatrix}$. Given this $W$ define $A, B$ via

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix} := \begin{bmatrix} D_{12} & -N_{12} \\ D_{22} & -N_{22} \end{bmatrix} W.$$

(d) By Assumption A2 we have that $B$ has full column rank on the imaginary axis, so there is a bistable $\bar{V}$ such that $\bar{V}^{\sim}\bar{V} = B^{\sim}B$. Redefine $A, B$ as

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix} := \begin{bmatrix} D_{12} & -N_{12} \\ D_{22} & -N_{22} \end{bmatrix} W \begin{bmatrix} I & 0 \\ 0 & \bar{V}^{-1} \end{bmatrix}.$$

(Now $B^{\sim}B = I$.)

(e) Let $\bar{F} := -\{B^{\sim}A\}_{+,\infty}$ and redefine $A, B$ once again

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix} := \begin{bmatrix} D_{12} & -N_{12} \\ D_{22} & -N_{22} \end{bmatrix} \underbrace{W \begin{bmatrix} I & 0 \\ 0 & \bar{V}^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ \bar{F} & I \end{bmatrix}}_{U:=}.$$

(Then $B^{\sim}A$ has no stable poles and is strictly proper.)

The equality (3) may now be written as

$$\begin{bmatrix} Q_K \\ I \end{bmatrix} := \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} \begin{bmatrix} I \\ Q \end{bmatrix}, \quad \begin{bmatrix} I \\ Q \end{bmatrix} := U^{-1} \begin{bmatrix} \bar{X} \\ \bar{Y} \end{bmatrix}. \qquad (5)$$

Note that $Q_K = A + BQ$. By Lemma 2.1 we therefore have that

$$\|Q_K\|_2^2 = \|A\|_2^2 + \|Q\|_2^2. \qquad (6)$$

Combination (4) and (6) yields $\|H\|_2^2 = \|N_{11}\|_2^2 + \|A\|_2^2 + \|Q\|_2^2$. It follows from (5) that $\left[\begin{smallmatrix} \bar{X} \\ \bar{Y} \end{smallmatrix}\right] = U\left[\begin{smallmatrix} I \\ Q \end{smallmatrix}\right]$, i.e., that $K := \bar{Y}\bar{X}^{-1} = (U_{21} + U_{22}Q)(U_{11} + U_{12}Q)^{-1}$. ∎

It is interesting to note that the method is not restricted to proper plants and controllers and they may in fact even be infinite-gain (i.e., $\bar{X}$ and $D$ may be singular). This is a feature of the method, a feature that is shares with most polynomial approaches, but not with most state-space approaches. In the event of infinite gain plants or controllers, the plant or controller can not be identified with the transfer matrix but as systems they are still be perfectly valid but then interpreted behavioristically, that is as the set of solutions of (2).

### 3.1 *Comparison with a state space approach*

If $G$ is proper then assumptions A1, A2, A3 are essentially equivalent to the usual state space assumptions (see e.g. (Zhou *et al.*, 1996, p.384-389)). It is quite straightforward to translate the state-space solution into that of Theorem 3.1 and conversely, to derive the state space solution from Theorem 3.1. There appears to be a definite advantage, however, with the state space approach over the frequency domain approach when it comes to computation: In state space the two steps (a) and (b) of the proof of Theorem 3.1 are performed in a single step and no explicit projection $F := -\{N_{11}N_{21}^{\sim}\}_{+,\infty}$ needs to be computed. Concretely, suppose $G(s)$ is proper and stable with minimal realization

$$G(s) = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (sI - A)^{-1} \begin{bmatrix} B_1 & B_2 \end{bmatrix} + \begin{bmatrix} 0 & E_{12} \\ E_{21} & 0 \end{bmatrix}$$

and suppose (to further simplify formulae) that $B_1E_{21}^* = 0$ and $E_{21}E_{21}^* = I$. For $D$ and $N$ we can take the joint realization

$$\begin{bmatrix} D_{11} & D_{12} & N_{11} & N_{12} \\ D_{21} & D_{22} & N_{21} & N_{22} \end{bmatrix} \overset{s}{=} \left[\begin{array}{c|cccc} A & 0 & 0 & B_1 & B_2 \\ \hline C_1 & I & 0 & 0 & E_{12} \\ C_2 & 0 & I & E_{21} & 0 \end{array}\right]$$

Now, the spectral factor $V$ of $VV^{\sim} = N_{21}N_{21}^{\sim}$ may be shown to be $V(s) = C_2(sI - A)^{-1}QC_2^* + I$, where $Q$ is the solution of the filter Riccati equation associated with $H_2$. In State space approaches it is now common to redefine the coprime factorization as [3]

[2] The $\{\cdot\}_{+,\infty}$ denotes the strictly proper stable part plus the polynomial part.

[3] In state space one does not actually work with coprime factorizations, but this is how one may interpret the state space formulae.

$$\begin{bmatrix} I & D_{12} & N_{11} & N_{12} \\ 0 & D_{22} & N_{21} & N_{22} \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|cccc} A - QC_2^*C_2 & 0 & -QC_2^* & B_1 - QC_2^*E_{21} & B_2 \\ \hline C_1 & I & 0 & 0 & E_{12} \\ C_2 & 0 & I & E_{21} & 0 \end{array}\right]$$

Now $N_{21}$ satisfies $N_{21}N_{21}^{\sim} = I$, that is, Step (a) has been performed. However, the above redefinition also has changed $N_{11}$. Surprisingly, $N_{11}N_{21}^{\sim}$ may be verified to be antistable. So the projection $F$ needed in Step (b) of the proof, $F := -\{N_{11}N_{21}^{\sim}\}_{+,\infty}$ is $F = 0$. That is, Step (b) is void. Similarly in the state space approach also the projection of Step (e) is void.

Is it possible to develop a frequency domain method that does not explicitly need projections (on top of the two spectral factorizations)?

## 4. POLYNOMIAL-STABLE SOLUTION

In the previous section we took coprime factorizations over the ring of stable transfer matrices. Such a factorization is useful if we want not only our closed loop poles in the open left-half plane but also want the map from all disturbances $w$, $v_1$, $v_2$ to all outputs $z, u, y$ proper. Properness is a property "at $s = \infty$". Unfortunately, this approach also *requires* a property at $s = \infty$: The assumptions A1 and A2 state that certain matrices are full rank at $s = \infty$. In many situations these two assumptions are not satisfied, for example in Wiener filtering with colored noise.

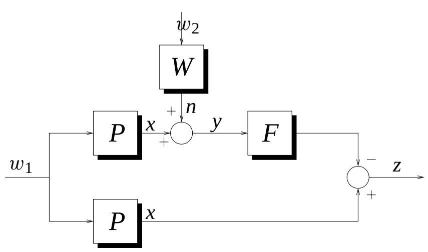


Fig. 3. Wiener filtering with colored noise.

*Example 4.1.* (Wiener filtering). Consider the system shown in Figure 3. A message signal $x$ is corrupted by colored noise $n = Ww_2$, driven by white noise $w_2$. To recover from $y$ the message $x$ one may want to minimize the $H_2$ norm of the map from $\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right]$ to $z$. If we set up the problem as a standard $H_2$ problem then we arrive at the generalized plant

$$G = \begin{bmatrix} P & 0 & -I \\ P & W & 0 \end{bmatrix}.$$

Now if both $P$ and $W$ are strictly proper (and they often are) then $G_{21} := \begin{bmatrix} P & W \end{bmatrix}$ is strictly proper and this violates the standard assumptions (Assumption A1 in our case), so the standard solution does not work directly. In this particular example the non-standardness can be easily remedied in the SISO case by redefining the output $y_0(s) = (s+1)^k y(s)$ where $k$ is relative degree of $\begin{bmatrix} P & W \end{bmatrix}$. From the then found filter $F_0$ we can form the optimal filter for the original problem, $F(s) = (s+1)^k F_0(s)$. □

A more systematic approach to handle non-standard $H_2$ problems can be done via a polynomial approach[4]. It allows to dispense with the assumptions at $s = \infty$, which are the assumptions we would like not to need.

Actually, instead of the polynomial factorizations we propose to take factorizations over the ring $\mathcal{Q}$ of polynomial-stable matrices,

$$\mathcal{Q} = \{\, Q \;:\; Q \text{ rational and } \{Q\}_- = 0\,\}.$$

For example, $1/(s+1)$ and $s^{10}/(s+1)$ are in $\mathcal{Q}$, but $s^2/(s-1)$ is not. The polynomial-stable matrices have slightly nicer properties than its subset of polynomial matrices.

Now let (1) denote polynomial-stable factorizations of $G$ and $K$. The assumptions that we now impose are practically the same as A1-A3, except that the point $s = \infty$ is left out: We assume that

**(B1)** $\begin{bmatrix} -N_{11} & D_{11} \\ -N_{21} & D_{21} \end{bmatrix}$ has full row rank on $j\mathbb{R}$,

**(B2)** $\begin{bmatrix} D_{21} & -N_{12} \\ D_{22} & -N_{22} \end{bmatrix}$ has full column rank on $j\mathbb{R}$,

**(B3)** The closed loop is stabilizable by some $K$.

Closed loop stability now means that all (finite) closed loop poles are in the open left-half plane. With the factorizations over $\mathcal{Q}$ in place of the factorizations over the stable matrices, Theorem 3.1 and its proof remain valid unaltered. For completeness:

*Theorem 4.2.* Suppose $G$ satisfies assumptions B1, B2, B3. Then there is a factorization over $\mathcal{Q}$ of $G$ such that $\left[\begin{smallmatrix} D_{11} \\ D_{21} \end{smallmatrix}\right] = \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$, and

(1) $N_{21}N_{21}^{\sim} = I$,
(2) $N_{11}N_{21}^{\sim}$ has no stable poles and is strictly proper.

Also there is then a bi-polynomial-stable $U$ such that $A$, $B$ defined as

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix} := \begin{bmatrix} D_{12} & -N_{12} \\ D_{22} & -N_{22} \end{bmatrix} U$$

satisfy

(3) $B^{\sim}B = I$,
(4) $B^{\sim}A$ has no stable poles and is strictly proper.

With these data, $K$ is stabilizing iff $K = (U_{21} + U_{22}Q)(U_{11} + U_{12}Q)^{-1}$ for some stable $Q$, and then

$$\|H\|_2^2 = \|N_{11}\|_2^2 + \|A\|_2^2 + \|Q\|_2^2.$$

In particular $\|H\|_2$ can be made finite if and only if both $N_{11}$ and $A$ are strictly proper, in which case the stabilizing controller $K$ that minimizes $\|H\|_2$ is given by $Q = 0$ (i.e., $K = U_{21}U_{11}^{-1}$). □

[4] In behavioral control problems, the presence of "disturbances" $v_i$ (see Fig. 2) is often not appropriate, and demanding that the map from these $v_i$ to $u$ and $y$ be proper is often not well motivated. Also in that case a polynomial approach is more natural than that based on factorizations over the ring of stable transfer matrices.

We apply the method to a non-standard example taken from (Kučera, 1996).

*Example 4.3.* ((Kučera, 1996), Ex. 1.6). Consider the generalized plant

$$G(s) = \begin{bmatrix} \frac{1}{s+2} & \frac{s-1}{s+2} \\ \frac{1}{s+2} & \frac{s-1}{s+2} \end{bmatrix}.$$

It fails to satisfy Assumption A1, but it satisfies B1, B2, B3, so Theorem 4.2 applies. As $G$ is stable we can take for $N$ and $D$,

$$\begin{bmatrix} D_{11} & D_{12} & N_{11} & N_{12} \\ D_{21} & D_{22} & N_{21} & N_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{1}{s+2} & \frac{s-1}{s+2} \\ 0 & 1 & \frac{1}{s+2} & \frac{s-1}{s+2} \end{bmatrix}.$$

We follow the steps of the proof of Theorem 3.1. Now, $V$ is a bi-polynomial-stable solution of $VV^{\sim} = N_{21}N_{21}^{\sim}$. As $N_{21} = 1/(s+2)$ we can simply take $V = N_{21} = 1/(s+2)$. Redefine accordingly (Step (a) of the proof),

$$\begin{bmatrix} D_{11} & D_{12} & N_{11} & N_{12} \\ D_{21} & D_{22} & N_{21} & N_{22} \end{bmatrix} := \begin{bmatrix} 1 & 0 & \frac{1}{s+2} & \frac{s-1}{s+2} \\ 0 & s+2 & 1 & s-1 \end{bmatrix}.$$

The next step is to compute $F = -\{N_{11}N_{21}^{\sim}\}_{+,\infty} = -\{\frac{1}{s+2}\}_{+,\infty} = \frac{-1}{s+2}$. With it redefine as in Step (b),

$$\begin{bmatrix} D_{11} & D_{12} & N_{11} & N_{12} \\ D_{21} & D_{22} & N_{21} & N_{22} \end{bmatrix} := \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & s+2 & 1 & s-1 \end{bmatrix}.$$

Next we have to find a bi-polynomial-stable $W$ such that $\begin{bmatrix} D_{22} & -N_{22} \end{bmatrix} W = \begin{bmatrix} s+2 & -(s-1) \end{bmatrix} W = \begin{bmatrix} 1 & 0 \end{bmatrix}$. We can take

$$W = \begin{bmatrix} 1/3 & s-1 \\ 1/3 & s+2 \end{bmatrix}.$$

In Step (c) we define $\begin{bmatrix} A & B \end{bmatrix} := \begin{bmatrix} D_{12} & -N_{12} \end{bmatrix} W = \begin{bmatrix} -1/3 & s-1 \end{bmatrix}$. Then, in Step (d) we have to compute bi-polynomial-stable $\bar{V}$ from $\bar{V}^{\sim}\bar{V} = B^{\sim}B = (1-s^2)$. We take $\bar{V} = s+1$. Then redefine as done in Step (d),

$$\begin{bmatrix} A & B \end{bmatrix} := \begin{bmatrix} A & B\bar{V}^{-1} \end{bmatrix} = \begin{bmatrix} 1/3 & \frac{s-1}{s+1} \end{bmatrix}.$$

Step (e) wants us to compute $\bar{F} := -\{B^{\sim}A\}_{+,\infty} = -\{\frac{-s-1}{-s+1}\frac{1}{3}\}_{+,\infty} = -1/3$. This, finally gives us $A := A + B\bar{F} = \frac{1}{3}\frac{2}{s+1}$ and

$$U := W \begin{bmatrix} 1 & 0 \\ 0 & \bar{V}^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \bar{F} & 1 \end{bmatrix} = \begin{bmatrix} \frac{2/3}{s+1} & \frac{s-1}{s+1} \\ \frac{-1/3}{s+1} & \frac{s+2}{s+1} \end{bmatrix}.$$

The optimal controller follows as $K = U_{21}U_{11}^{-1} = -1/2$, and all sub-level stabilizing controllers are parameterized by

$$K = \frac{U_{21} + U_{22}Q}{U_{11} + U_{12}Q} = \frac{-1 + 3(s+2)Q}{2 + 3(s-1)Q}, \quad Q \text{ stable.}$$

The closed loop transfer matrix $H$ has 2-norm $\|H\|_2 = \sqrt{\|N_{11}\|_2^2 + \|A\|_2^2 + \|Q\|_2^2} = \sqrt{0 + 1/3 + \|Q\|_2^2}$. □

There has been a polynomial method for the non-standard $H_\infty$ control problem for quite a few years, see e.g., (Kwakernaak, 1993; Meinsma, 1993), and for that problem the results are more satisfactory than that for $H_2$. For example, from the polynomial $H_\infty$ solution it is easy to show (also in the non-standard case) that the McMillan degree of the controller is at most that of the generalized plant $G$ (Meinsma, 1993). Whether the same is true for the $H_2$ problem is not easy to answer, at least not from our results. In the state space setting it is however trivial, but of course that only applies to a restrictive special case.

## 5. A FREQUENCY DOMAIN LQ PROBLEM

The LQ problem is normally formulated in time-domain in terms of an initial state $x(0)$ of some state-space realization. In this section we alternatively formulate the LQ problem without reference to states and we solve it in frequency domain terms. It will be convenient to denote the unit step function by $\mathbb{1}(t)$ and to use sub scripts $-$ and $+$ to denote past and future of a signal,

$$v_-(t) = v(t)\mathbb{1}(-t), \qquad v_+(t) = v(t)\mathbb{1}(t).$$

*Definition 5.1.* Consider a system $y = Gu$. The *LQ-problem* is to minimize $\int_0^\infty |u_+(t)|^2 + |y_+(t)|^2\,dt$ over the futures $(u_+, y_+)$ given a past $(u_-, y_-)$. □

The link with LQ will be clear. Some differences and other things to note are:

- No state is needed,
- The solution generally depends on the past $(u_-, y_-)$,
- It is not (yet) a feedback control problem.

Now let $G = NM^{-1}$ be a normalized coprime factorization of $G$, that is, $N$ and $M$ are coprime and $N^{\sim}N + M^{\sim}M = I$. We can represent the system $y = Gu$ by

$$y = N\underbrace{M^{-1}u}_{v}$$

and also by an image representation

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} N \\ M \end{bmatrix} v. \tag{7}$$

As $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$ is stable it is direct that the past of $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$ is determined by the past of $v$. It is interesting to note that coprimeness of $(N, M)$ implies the converse as well: The past $\left[\begin{smallmatrix} y_- \\ u_- \end{smallmatrix}\right]$ determines the past $v_-$ of $v$. Indeed, as $(N, M)$ are coprime there exist stable $(X, Z)$ for which $XN + ZM = I$. Therefore $v = Xy + Zu$ and in particular this shows that the past $v_-$ of $v$ is determined by the past of $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$.

As the coprime factorization was taken normalized (i.e, $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$ is inner) we actually solved the LQ-problem:

*Theorem 5.2.* Consider a system $y = NM^{-1}u$ with $NM^{-1}$ normalized coprime. Given the past $\left[\begin{smallmatrix} y_- \\ u_- \end{smallmatrix}\right]$, the LQ-optimal trajectory is $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] = \left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]v_-$, where $v_-$ is determined by the past of $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$ as shown above.

Moreover the optimal cost $\left\|\left[\begin{smallmatrix} y_+ \\ u_+ \end{smallmatrix}\right]\right\|_2^2$ equals $\|v_-\|_2^2 - \left\|\left[\begin{smallmatrix} y_- \\ u_- \end{smallmatrix}\right]\right\|_2^2$.

**PROOF.** Recall that there is a bijection from $v_-$ to $\left[\begin{smallmatrix} y_- \\ u_- \end{smallmatrix}\right]$ in (7).

Minimizing $\left\|\left[\begin{smallmatrix} y_+ \\ u_+ \end{smallmatrix}\right]\right\|_2^2$ is the same as minimizing $\left\|\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]\right\|_2^2$ as they differ a given constant $\left\|\left[\begin{smallmatrix} y_- \\ u_- \end{smallmatrix}\right]\right\|_2^2$. But, as $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$ is inner, we have that $\left\|\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]\right\|_2 = \|v\|_2$. Therefore

$$\min_{y=Gu,\ \text{given}\left[\begin{smallmatrix} y_- \\ u_- \end{smallmatrix}\right]} \left\|\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]\right\|_2^2 = \min_{v,\ \text{given}\ v_-} \|v\|_2^2 = \|v_-\|_2^2.$$

That is, setting the future $v_+$ of $v$ to zero minimizes the cost and the minimal cost $\left\|\left[\begin{smallmatrix} y_+ \\ u_+ \end{smallmatrix}\right]\right\|_2^2$ then equals $\|v_-\|_2^2 - \left\|\left[\begin{smallmatrix} y_- \\ u_- \end{smallmatrix}\right]\right\|_2^2$. ■

*Example 5.3.* Suppose $G(s) = 1/s$. Then

$$G = NM^{-1}, \qquad N(s) = \frac{1}{s+1}, \quad M(s) = \frac{s}{s+1}$$

is a normalized coprime factorization of $G$. By Theorem 5.2, $(y, u)$ is LQ-optimal if and only if (in frequency domain)

$$\begin{bmatrix} y(s) \\ u(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{s+1} \\ \frac{s}{s+1} \end{bmatrix} v_-(s).$$

In particular the future $(u_+, y_+)$ equals

$$\begin{bmatrix} y_+(s) \\ u_+(s) \end{bmatrix} = \pi_+ \begin{bmatrix} \frac{1}{s+1} v_-(s) \\ \frac{s}{s+1} v_-(s) \end{bmatrix} = \begin{bmatrix} \frac{v_-(-1)}{s+1} \\ \frac{-v_-(-1)}{s+1} \end{bmatrix}.$$

The LQ-optimal trajectories thus have the form $y_+(t) = -u_+(t) = \alpha\, \mathrm{e}^{-t}$ for some $\alpha$. □

*Example 5.4.* When the transfer matrix $G$ is given by a state space realization $G(s) = C(sI - A)^{-1}B + D$ then we recover the ubiquitous state feedback law.

It is well known that if the realization of $G$ is stabilizable and detectable that then the normalized coprime factors $N$ and $M$ have the joint realization

$$\begin{bmatrix} N \\ M \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|c} A - BF & BW_\infty^{-1} \\ \hline C - DF & DW_\infty^{-1} \\ -F & W_\infty^{-1} \end{array}\right],$$

where $P$ is the stabilizing solution of $PA + A^T P - (PB + C^T D)(I + D^T D)^{-1}(D^T C + B^T P) + C^T C = 0$ and $F := (I + D^T D)^{-1}(D^T C + B^T P)$ and $W_\infty$ any square (constant) solution of $W_\infty^T W_\infty = I + D^T D$.

According to Theorem 5.2 the LQ-optimal trajectories satisfy $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] = \left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] v_-$, which for positive time (i.e., where $v_-(t)$ is zero) gives us

$$\begin{bmatrix} \dot{x} \\ y \\ u \end{bmatrix} = \begin{bmatrix} A - BF \\ C - DF \\ -F \end{bmatrix} x.$$

We immediately recognize here that the LQ-optimal trajectories may be implemented by the state feedback law $u = -Fx$. □

The formulation of the LQ problem assumes that the past is given. As a result, the LQ-optimal trajectory $\left[\begin{smallmatrix} y_+ \\ u_+ \end{smallmatrix}\right]$ also depends on the past. Indeed, that is the case. On the other hand, Theorem 5.2 shows that $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$ is LQ-optimal if and only if $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in Mv_-$, for some past $v_-$, and this makes no reference to the specific past of $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$. For example, we found in Example 5.3 that $\left[\begin{smallmatrix} y_+ \\ u_+ \end{smallmatrix}\right]$ is LQ-optimal iff $y_+(t) = -u_+(t) = \alpha\, \mathrm{e}^{-t}$ for some $\alpha$. The *value* of $\alpha$ is of course determined by the past $\left[\begin{smallmatrix} y_- \\ u_- \end{smallmatrix}\right]$ but for *any* $\alpha$ the trajectory $y_+(t) = -u_+(t) = \alpha\, \mathrm{e}^{-t}$ is LQ-optimal, albeit for different pasts.

*Example 5.5.* (Behaviors). The so formulated LQ-problem in fact makes no distinction between inputs and outputs. Let $w$ denote the *external signal*, of the system, which in the input-output setting would mean $w = \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$. The LQ-problem is now to minimize the future energy $\|w_+\|_2$ over all future continuations $w_+$ of a given past $w_-$ subject to the system equations. If we can find an image representation of the system [5]

$$w = Mv, \quad (M \text{ inner}, ZM = I \text{ for some stable } Z)$$

then, as before, it may be shown that $w_+$ is LQ-optimal if and only if $w = Mv_-$ for some past $v_-$, i.e., if and only if $w_+(t) \in C\mathrm{e}^{At}\mathbb{R}^n$, where $C$ and $A$ come from a minimal realization of $M(s) = C(sI - A)^{-1}B + D$. □

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[5] (Weiland, 1991) showed that such a representation always exists if we restrict $w$ to be in $L_2$ and the system is described by a set of ordinary differential equations with constant coefficients, $\sum_{k,l} a_{rkl} \frac{d^l}{dt^l} w_k(t) = 0\ \forall r$.