On Approximating Restricted Cycle Covers*

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Abstract

A cycle cover of a graph is a set of cycles such that every vertex is part of exactly one cycle. An L-cycle cover is a cycle cover in which the length of every cycle is in the set L. The weight of a cycle cover of an edge-weighted graph is the sum of the weights of its edges.

We come close to settling the complexity and approximability of computing L-cycle covers. On the one hand, we show that for almost all L, computing L-cycle covers of maximum weight in directed and undirected graphs is APX-hard. Most of our hardness results hold even if the edge weights are restricted to zero and one.

On the other hand, we show that the problem of computing L-cycle covers of maximum weight can be approximated within a factor of 2 for undirected graphs and within a factor of 8/3 in the case of directed graphs. This holds for arbitrary sets L.

1 Introduction

A cycle cover of a graph is a spanning subgraph that consists solely of cycles such that every vertex is part of exactly one cycle. Cycle covers play an important role in the design of approximation algorithms for the traveling salesman problem [4, 6, 7, 10–13, 23], the shortest common superstring problem [9, 30], and vehicle routing problems [19].

In contrast to Hamiltonian cycles, which are special cases of cycle covers, cycle covers of maximum weight can be computed efficiently. This is exploited in the aforementioned approximation algorithms, which usually start by computing an initial cycle cover and then join cycles to obtain a Hamiltonian cycle. This technique is called *subtour patching* [16].

Short cycles in a cycle cover limit the approximation ratios achieved by such algorithms. In general, the longer the cycles in the initial cover, the better the approximation ratio. Thus, we are interested in computing cycle covers that do not contain short cycles. Moreover, there are approximation algorithms that perform particularly well if the cycle covers computed do not contain cycles of odd length [6]. Finally, some vehicle routing problems [19] require covering vertices with cycles of bounded length.

Therefore, we consider *restricted cycle covers*, where cycles of certain lengths are ruled out a priori: For $L \subseteq \mathbb{N}$, an *L-cycle cover* is a cycle cover in which the length of each

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cycle is in L. To fathom the possibility of designing approximation algorithms based on computing cycle covers, we aim to characterize the sets L for which L-cycle covers of maximum weight can be computed, or at least well approximated, efficiently.

Beyond being a basic tool for approximation algorithms, cycle covers are interesting in their own right. Matching theory and graph factorization are important topics in graph theory. The classical matching problem is the problem of finding one-factors, i. e., spanning subgraphs each vertex of which is incident to exactly one edge. Cycle covers of undirected graphs are also known as two-factors because every vertex is incident to exactly two edges. A considerable amount of research has been done on structural properties of graph factors and on the complexity of finding graph factors (cf. Lovász and Plummer [24] and Schrijver [29]). In particular, the complexity of finding restricted two-factors, i. e., L-cycle covers in undirected graphs, has been investigated, and Hell et al. [22] showed that finding L-cycle covers in undirected graphs is NP-hard for almost all L. However, almost nothing is known so far about the complexity of finding directed L-cycle covers.

1.1 Preliminaries

Let $G = (V, E)$ be a graph with vertex set V and edge set E. If G is undirected, then a **cycle cover** of G is a subset $C \subseteq E$ of the edges of G such that all vertices in V are incident to exactly two edges in C. If G is a directed graph, then a cycle cover of G is a subset $C \subseteq E$ such that all vertices are incident to exactly one incoming and one outgoing edge in C. Thus, the graph (V, C) consists solely of vertex-disjoint cycles. The length of a cycle is the number of edges it consists of. Since we do not allow self-loops or multiple edges, the shortest cycles of undirected and directed graphs are of length three and two, respectively.

We call a cycle of length λ a λ-cycle for short. Cycles of even or odd length will simply be called even or odd cycles, respectively.

An ***L*-cycle cover** of an undirected graph is a cycle cover in which the length of every cycle is in $L \subseteq \mathcal{U} = \{3, 4, 5, \ldots\}$. An L-cycle cover of a directed graph is analogously defined except that $L \subseteq \mathcal{D} = \{2, 3, 4, \ldots\}$. A ***k*-cycle cover** is a $\{k, k+1, \ldots\}$-cycle cover. In the following, let $\overline{L} = \mathcal{U} \setminus L$ in the case of undirected graphs and $\overline{L} = \mathcal{D} \setminus L$ in the case of directed graphs (whether we consider undirected or directed cycle covers will be clear from the context).

Given edge weights $w : E \to \mathbb{N}$, the **weight $\boldsymbol{w(C)}$** of a subset $C \subseteq E$ of the edges of G is $w(C) = \sum_{e \in C} w(e)$. In particular, this defines the weight of a cycle cover since we view cycle covers as sets of edges. Let $U \subseteq V$ be any subset of the vertices of G. The **internal edges of $\boldsymbol{U}$** are all edges of G that have both vertices in U. We denote by $\boldsymbol{w_U(C)}$ the sum of the weights of all internal edges of U that are also contained in C. The **external edges at $\boldsymbol{U}$** are all edges of G with exactly one vertex in U.

For $L \subseteq \mathcal{U}$, ***L*-UCC** is the decision problem whether an undirected graph contains an L-cycle cover as spanning subgraph.

Max-*L*-UCC(0,1) is the following optimization problem: Given an undirected complete graph with edge weights zero and one, find an L-cycle cover of maximum weight. We can also consider the graph as being not complete and without edge weights. Then we try to find an L-cycle cover with a minimum number of "non-edges" ("non-edges" correspond to weight zero edges, edges to weight one edges), i. e., the L-cycle cover should contain as many edges as possible. Thus, Max-L-UCC(0,1) generalizes L-UCC.

Max-*L*-UCC is the problem of finding L-cycle covers of maximum weight in graphs with arbitrary non-negative edge weights.

For $k \in \mathcal{U}$, **k-UCC**, **Max-k-UCC(0,1)**, and **Max-k-UCC** are defined like L-UCC, Max-L-UCC(0,1), and Max-L-UCC except that k-cycle covers rather than L-cycle covers are sought.

The problems **L-DCC**, **Max-L-DCC(0,1)**, and **Max-L-DCC** as well as **k-DCC**, **Max-k-DCC(0,1)**, and **Max-k-DCC** are defined for directed graphs like their undirected counterparts except that $L \subseteq \mathcal{D}$ and $k \in \mathcal{D}$.

An instance of **Min-Vertex-Cover(λ)** is an undirected λ-regular graph $H = (X, F)$, i.e., every vertex in X is incident to exactly λ edges. A vertex cover of H is a subset $\tilde{X} \subseteq X$ such that at least one vertex of every edge in F is in $\tilde{X}$. The aim is to find a subset $\tilde{X} \subseteq X$ of minimum cardinality. Min-Vertex-Cover(λ) is APX-complete for $\lambda \geq 3$ as follows from results by Alimonti and Kann [2] as well as Chlebík and Chlebíková [14].

An instance of **λ-XC** (exact cover by λ-sets) is a tuple (X, F) where X is a finite set and F is a collection of subsets of X, each of cardinality λ. The question is whether there exists a sub-collection $\tilde{F} \subseteq F$ such that for every $x \in X$ there is a unique $a \in \tilde{F}$ with $x \in a$. For $\lambda \geq 3$, λ-XC is NP-complete [15, SP2].

Let Π be an optimization problem, and let I be its set of instances. For an instance $X \in I$, let $\operatorname{opt}(X)$ denote the weight of an optimum solution. We say that Π can be approximated with an approximation ratio of $\alpha \geq 1$ if there exist a polynomial-time algorithm that, for every instance $X \in I$, computes a solution Y of X whose weight $w(Y, X)$ is at most a factor of α away from $\operatorname{opt}(X)$. This means that $w(Y, X) \leq \alpha \cdot \operatorname{opt}(X)$ if Π is a minimization problem and $w(Y, X) \geq \operatorname{opt}(X)/\alpha$ if Π is a maximization problem [3, Definition 3.6].

1.2 Previous Results

Max-$\mathcal{U}$-UCC, and thus $\mathcal{U}$-UCC and Max-$\mathcal{U}$-UCC(0,1), can be solved in polynomial time via Tutte's reduction to the classical perfect matching problem [24, Section 10.1]. Hartvigsen presented a polynomial-time algorithm that can be used to decide 4-UCC in polynomial time [17]. Furthermore, it can be adapted to solve Max-4-UCC(0,1) as well.

Max-k-UCC admits a simple factor 3/2 approximation for all k: Compute a maximum weight cycle cover, break the lightest edge of each cycle, and join the paths thus obtained to a Hamiltonian cycle. Unfortunately, this algorithm cannot be generalized to work for Max-L-UCC for general L. For the problem of computing k-cycle covers of minimum weight in graphs with edge weights one and two, there exists a factor 7/6 approximation algorithm for all k [8]. Hassin and Rubinstein [20,21] devised a randomized approximation algorithm for Max-{3}-UCC that achieves an approximation ratio of $83/43 + \epsilon$.

Hell et al. [22] proved that L-UCC is NP-hard for $\overline{L} \not\subseteq \{3, 4\}$. For $k \geq 7$, Max-k-UCC(0,1) and Max-k-UCC are APX-complete [5]. Vornberger showed that Max-5-UCC is NP-hard [31].

The directed cycle cover problems $\mathcal{D}$-DCC, Max-$\mathcal{D}$-DCC(0,1), and Max-$\mathcal{D}$-DCC can be solved in polynomial time by reduction to the maximum weight perfect matching problem in bipartite graphs [1, Chapter 12]. But already 3-DCC is NP-complete [15]. Max-k-DCC(0,1) and Max-k-DCC are APX-complete for all $k \geq 3$ [5].

Similar to the factor 3/2 approximation algorithm for undirected cycle covers, Max-k-DCC has a simple factor 2 approximation algorithm for all k: Compute a maximum weight cycle cover, break the lightest edge of every cycle, and join the cycles to obtain a Hamiltonian cycle. Again, this algorithm cannot be generalized to work for arbitrary L. There is a factor 4/3 approximation algorithm for Max-3-DCC [7] and a factor 3/2 approximation algorithm for Max-k-DCC(0,1) for $k \geq 3$ [5].

The complexity of finding L-cycle covers in undirected graphs seems to be well understood. However, hardly anything is known about the complexity of L-cycle covers in directed graphs and about the approximability of L-cycle covers in both undirected and directed graphs.

1.3 Our Results

We prove that Max-L-UCC(0,1) is APX-hard for all L with $\overline{L} \not\subseteq \{3,4\}$ (Section 2.2) and that Max-L-UCC is APX-hard if $\overline{L} \not\subseteq \{3\}$ (Section 2.3). The hardness results for Max-L-UCC hold even if we allow only the edge weights zero, one, and two.

We show a dichotomy for directed graphs: For all L with $L \neq \{2\}$ and $L \neq \mathcal{D}$, L-DCC is NP-hard and Max-L-DCC(0,1) and Max-L-DCC are APX-hard (Section 2.5), while all three problems are solvable in polynomial time if $L = \{2\}$ or $L = \mathcal{D}$.

The hardness results for Max-L-UCC(0,1) and Max-L-DCC(0,1) carry over to the problem of computing L-cycle covers of minimum weight in graphs restricted to edge weights one and two. The hardness results for Max-L-UCC for $\overline{L} = \{3,4\}$ and $\overline{L} = \{4\}$ carry over to the problem of computing L-cycle covers of minimum weight where the edge weights are required to fulfill the triangle inequality.

To show the hardness of directed cycle covers, we show that certain kinds of graphs, called *L-clamps*, exist for non-empty $L \subseteq \mathcal{D}$ if and only if $L \neq \mathcal{D}$ (Theorem 2.10). This graph-theoretical result might be of independent interest.

Finally, we devise approximation algorithms for Max-L-UCC and Max-L-DCC that achieve ratios of 2 and 8/3, respectively (Section 3). Both algorithms work for all sets L.

2 The Hardness of Approximating L-Cycle Covers

2.1 Clamps and Gadgets

To begin the hardness proofs, we introduce *clamps*, which were defined by Hell et al. [22]. Clamps are crucial for our hardness proof.

Let $K = (U, E)$ be an undirected graph, and let $u, v \in U$ be two vertices of K, which we call the **connectors** of K. We denote by K_{-u} and K_{-v} the graphs obtained from K by deleting u and v, respectively, and their incident edges. K_{-u-v} is obtained from K by deleting both u and v. For $k \in \mathbb{N}$, K^k is the following graph: Let $y_1, \ldots, y_k \notin U$ be new vertices, add edges $\{u, y_1\}$, $\{y_i, y_{i+1}\}$ for $1 \leq i \leq k-1$, and $\{y_k, v\}$. For $k = 0$, we directly connect u to v.

Let $L \subseteq \mathcal{U}$. The graph K is called an ***L*-clamp** if the following properties hold:

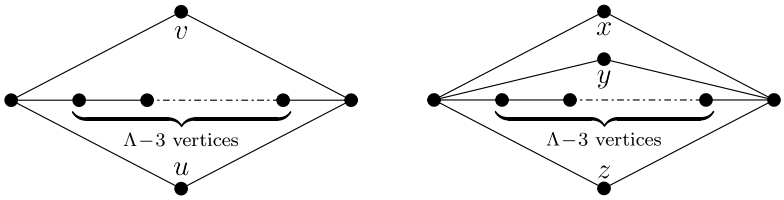
1. Both K_{-u} and K_{-v} contain an L-cycle cover.

2. Neither K nor K_{-u-v} nor K^k for any $k \in \mathbb{N}$ contains an L-cycle cover.

Figure 1(a) shows an example of an L-clamp for a set L with $\Lambda = \max(L)$. Hell et al. [22] proved the following result which we will exploit for our reduction.

Lemma 2.1 (Hell et al. [22])**.** *Let $L \subseteq \mathcal{U}$ be non-empty. Then there exists an L-clamp if and only if $\overline{L} \not\subseteq \{3,4\}$.*

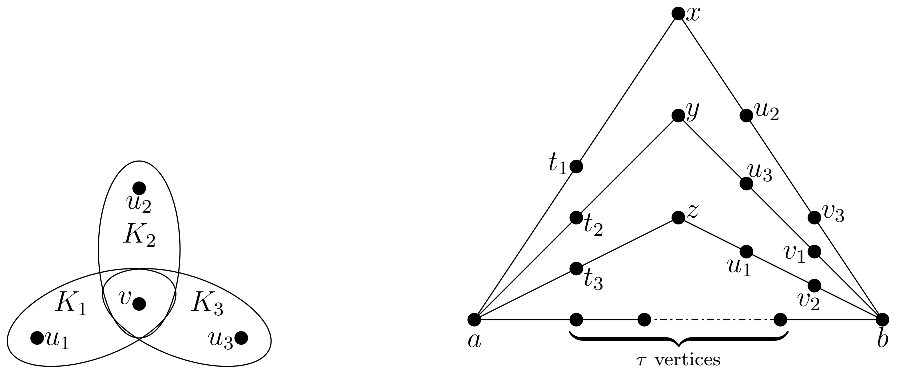
Let G be a graph with vertex set V and $U \subseteq V$. We say that the vertex set U is an L-clamp with connectors $u, v \in U$ in G if the subgraph of G induced by U is an L-clamp and the only external edges of U are incident to u or v.



(a) L-clamp.

(b) L-gadget.

Figure 1: An L-clamp and an L-gadget for a set L with $\max(L) = \Lambda$.



(a) A triple L-clamp with connectors u_1, u_2, u_3. The connectors of L-clamp K_i are u_i and v.

(b) An L-gadget with connectors x, y, z. The connectors of triple L-clamps T_i are t_i, u_i, v_i. For legibility, the triple L-clamps are not shown explicitly but only their connectors.

Figure 2: A triple L-clamp and an L-gadget.

Let us fix some technical terms. For this purpose, let C be a subset of the edges of G. (In particular, C can be a cycle cover of G.) For any $V' \subseteq V$, we say that **V' is isolated in C** if there is no edge in C connecting V' to $V \setminus V'$. If C is a cycle cover, then this means that all cycles of C traverse either only nodes of V' or only nodes of $V \setminus V'$. We say that the L-clamp **U absorbs u and expels v** if $U \setminus \{v\}$ is isolated in C. This means that each cycle of C traverses either only vertices in $(V \setminus U) \cup \{v\}$ or only vertices in $U \setminus \{v\}$ (which includes u). Analogously, **U absorbs v and expels u** if $U \setminus \{u\}$ is isolated in C.

An L-clamp implements an exclusive-or of u and v: In every L-cycle cover, exactly one of them is absorbed, the other one is expelled. For our purpose of reducing from Min-Vertex-Cover(λ), we need a one-out-of-three behavior. A graph K is called an **L-gadget with connectors x, y, z** if the following property is fulfilled: Let G be an arbitrary graph that contains K as a subgraph such that only x, y, and z are incident to external edges. Then in all L-cycle covers C of G, exactly two of K's connectors are expelled while the third one is absorbed. To put it another way: Either K_{-x-y} or K_{-x-z} or K_{-y-z} is isolated in C.

For finite sets L, we obtain an L-gadget, shown in Figure 1(b), by equipping the L-clamp of Figure 1(a) with an additional connector.

For infinite sets L, we first build an intermediate subgraph. A **triple L-clamp** is built from three L-clamps and has three connectors u_1, u_2, u_3. Figure 2(a) shows the construction. Triple L-clamps show a two-out-of-three behavior: Only one connector will be expelled, the other two will be absorbed. More precisely: One of the three clamps has to absorb v. The other two absorb their connectors u_i, which are also connectors of the triple clamp.

Now we are prepared to build L-gadgets for infinite sets L. These graphs are built from three triple L-clamps T_1, T_2, and T_3, where T_i has connectors u_i, v_i, t_i. Figure 2(b) shows the L-gadget. Since L is infinite, there exists a $\tau \geq 1$ with $\tau + 6 \in L$. Let us argue why the L-gadget behaves as claimed. For this purpose, let C be an arbitrary L-cycle cover of G, where G contains the L-gadget as a subgraph. First, we observe that all $\tau + 2$ vertices of the path connecting a to b must be on the same cycle c in C. The only other vertices to which a is incident are t_1, t_2, and t_3. By symmetry, we assume that t_1 lies also in c. Therefore, T_1 absorbs u_1 and v_1. Hence, v_2 and u_3 are absorbed by T_2 and T_3, respectively, and c runs through x, u_2, v_3 back to b to form a $(\tau + 6)$-cycle. Thus, x is absorbed by the gadget. T_2 expels u_2 and absorbs u_3, while T_3 expels v_3 and absorbs v_2. Hence, the gadget expels y and z as claimed. The other two cases are symmetric.

To conclude this section about clamps, we transfer the notion of L-gadgets to complete graphs with edge weights zero and one and prove some properties. In Section 2.3, we will generalize the notion of L-gadgets to graphs with arbitrary edge weights.

The transformation to graphs with edge weights zero and one is made in the obvious way: Let G be an undirected complete graph with vertex set V and edge weights zero and one. Let $U \subseteq V$. We say that U is an L-gadget with connectors $x, y, z \in U$ if the subgraph of G induced by U restricted to the edges of weight one is an L-gadget with connectors x, y, z.

Let σ be the number of vertices of an L-gadget U with connectors x, y, and z. Let C be a subset of the edges of G (in particular, C can be a cycle cover). We call U **healthy in C** if U absorbs x, y, or z, expels the other two connectors, and $w_U(C) = \sigma - 2$. Since the edge weighted graph is complete, L-cycle may traverse L-gadgets arbitrarily. The following lemma shows that we cannot gain weight by not traversing them healthily.

Lemma 2.2. *Let G be an undirected graph with vertex set V and edge weights zero and one, and let $U \subseteq V$ be an L-gadget with connectors x, y, z. Let C be an arbitrary L-cycle cover of G and $|U| = \sigma$. Then the following properties hold:*

1. *$w_U(C) \leq \sigma - 1$.*
2. *If there are 2α external edges at U in C, i. e., edges with exactly one endpoint in U, then $w_U(C) \leq \sigma - \alpha$.*
3. *Assume that U absorbs exactly one of x, y, or z. Then there exists an L-cycle cover $\tilde{C}$ that differs from C only in the internal edges of U and has $w_U(\tilde{C}) = \sigma - 2$.*
4. *Assume that there are two external edges at U in C that are incident to two different connectors. Then $w_U(C) \leq \sigma - 2$.*

Proof. If $w_U(C) = \sigma$ was true, then U would contain an L-cycle cover consisting solely of weight one edges since $|U| = \sigma$. This would contradict U being an L-gadget.

The second claim follows immediately from $|U| = \sigma$ and the fact that every vertex is incident to exactly two edges in a cycle cover.

Assume without loss of generality that U absorbs x and expels y and z. Since U is an L-gadget, $U \setminus \{y, z\}$ contains an L-cycle cover consisting of $\sigma - 1$ weight one edges, which proves the third claim.

The fourth claim remains to be proved. If there are more than two external edges at U in C, we have at least four external edges and thus $w_U(C) \leq \sigma - 2$. So assume that there are exactly two external edges at U in C incident to, say, x and y. We have $\sigma - 1$ internal edges of U in C. If all of them had weight one, this would contradict the property that in an unweighted L-gadget always $U \setminus \{x, y\}$, $U \setminus \{x, z\}$, or $U \setminus \{y, z\}$ is isolated. $\square$

2.2 The Reduction for Undirected Graphs

The notion of L-reductions was introduced by Papadimitriou and Yannakakis [27] (cf. Ausiello et al. [3, Definition 8.4]). L-reductions can be used to show the APX-hardness of optimization problems. We present an L-reduction from Min-Vertex-Cover(λ) to show the inapproximability of Max-L-UCC(0,1) for $\overline{L} \not\subseteq \{3,4\}$. The inapproximability of Max-L-UCC for $\overline{L} \not\subseteq \{3\}$ and Max-L-DCC(0,1) for $L \neq \{2\}$ and $L \neq \mathcal{D}$ will be shown in subsequent sections.

Let $L \subseteq \mathcal{U}$ be non-empty with $\overline{L} \not\subseteq \{3,4\}$. Thus, L-gadgets exist and we fix one as in the previous section. Let $\lambda = \min(L)$. (This choice is arbitrary. We could choose any number in L.) We will reduce Min-Vertex-Cover(λ) to Max-L-UCC(0,1). Min-Vertex-Cover(λ) is APX-complete since $\lambda \geq 3$.

Let $H = (X, F)$ be an instance of Min-Vertex-Cover(λ) with $|X| = n$ vertices and $|F| = m = \lambda n/2$ edges. Our instance G for Max-L-UCC(0,1) consists of λ subgraphs $G_1, \ldots, G_\lambda$, each containing σm vertices, where σ is the number of vertices of the L-gadget. We start by describing G_1. Then we state the differences between G_1 and $G_2, \ldots, G_\lambda$ and say to which external edges of $G_1, \ldots, G_\lambda$ weight one is assigned.

Let $a = \{x, y\} \in F$ be any edge of H. We construct an L-gadget F_a for a that has connectors x_a^1, y_a^1 and z_a^1. We call F_a an **edge gadget**.

Now let $x \in X$ be any vertex of H and let $a_1, \ldots, a_\lambda \in F$ be the λ edges that are incident to x. We connect the vertices $x_{a_1}^1, \ldots, x_{a_\lambda}^1$ to form a path by assigning weight one to the edges $\{x_{a_\eta}^1, x_{a_{\eta+1}}^1\}$ for $\eta \in \{1, \ldots, \lambda-1\}$. Together with edge $\{x_{a_\lambda}^1, x_{a_1}^1\}$, these edges form a cycle of length $\lambda \in L$, but note that $w(\{x_{a_\lambda}^1, x_{a_1}^1\}) = 0$. These λ edges are called the **junctions of $\boldsymbol{x}$**. The **junctions at $\boldsymbol{F_a}$** for some $a = \{x, y\} \in F$ are the junctions of x and y that are incident to F_a. Overall, the graph G_1 consists of σm vertices since every edge gadget consists of σ vertices.

The graphs $G_2, \ldots, G_\lambda$ are almost exact copies of G_1. The graph G_ξ ($\xi \in \{2, \ldots, \lambda\}$) consists of L-gadgets with connectors x_a^ξ, y_a^ξ, and z_a^ξ for each edge $a = \{x, y\} \in F$, just as above. The edge weights are also identical with the single exception that the edge $\{x_{a_\lambda}^\xi, x_{a_1}^\xi\}$ also has weight one. Note that we use the term "edge gadget" only for the subgraphs F_a of G_1 defined above although almost the same subgraphs occur in $G_2, \ldots, G_\lambda$ as well. Similarly, the term "junction" refers only to edges in G_1.

Finally, we describe how to connect $G_1, \ldots, G_\lambda$ with each other. For every edge $a \in F$, there are λ vertices $z_a^1, \ldots, z_a^\lambda$. These are connected to form a cycle consisting solely of weight one edges, i. e., we assign weight one to all edges $\{z_a^\xi, z_a^{\xi+1}\}$ for $\xi \in \{1, \ldots, \lambda - 1\}$ and to $\{z_a^\lambda, z_a^1\}$. Figure 3 shows an example of the whole construction from the viewpoint of a single vertex.

Edges with both vertices in the same gadget are called **internal edges**. Besides junctions and internal edges, the third kind of edges are the ***z*-edges** of F_a for $a \in F$, which are the two edges $\{z_a^1, z_a^2\}$ and $\{z_a^1, z_a^\lambda\}$. The fourth kind of edges are **illegal edges**, which are edges that are not junctions but connect any two vertices of two different gadgets. The z-edges, however, are not illegal. Edges within $G_2, \ldots, G_\lambda$ as well as edges connecting G_ξ to $G_{\xi'}$ for $\xi, \xi' \geq 2$ have no special name.

We define the following terms for arbitrary subsets C of the edges of the graph G thus constructed, which includes the case of C being a cycle cover. Let $a = \{x, y\} \in F$ be an arbitrary edge of H. We say that ***C* legally connects *F*$_a$** if the following properties are fulfilled:

- C contains either two or four of the junctions at F_a and no illegal edges incident to F_a.

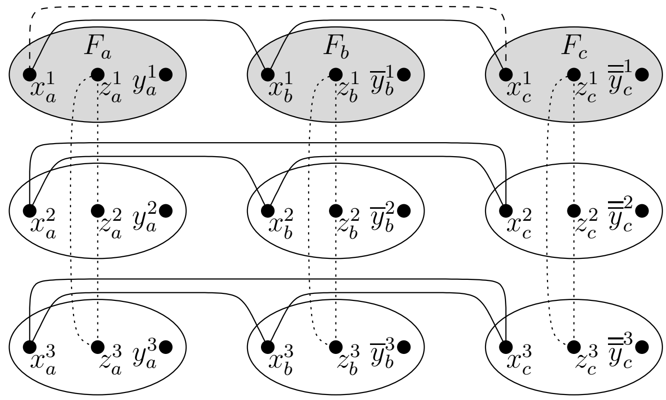


Figure 3: The construction for $x \in X$ incident to $a = \{x, y\}, b = \{x, \overline{y}\}, c = \{x, \overline{\overline{y}}\} \in F$ for $\lambda = 3$. F_a, F_b, and F_c are grey. The three ellipses in the second and third row build G_2 and G_3, respectively. The cycles connecting the z-vertices are dotted. The junctions of x and their copies are solid, except for $\{x_c^1, x_a^1\}$, which has weight zero and is dashed.

- If C contains exactly two junctions at F_a, then these belong to the same vertex and the two z-edges at F_a are contained in C.
- If C contains four junctions at F_a, then C does not contain the z-edges at F_a.

We call C **legal** if C legally connects all gadgets. If $\tilde{C}$ is a legal L-cycle cover, then for all $x \in X$ either all junctions of x or no junction of x is in $\tilde{C}$. From a legal L-cycle cover $\tilde{C}$, we obtain the subset $\tilde{X} = \{x \mid \text{the junctions of } x \text{ are in } \tilde{C}\} \subseteq X$. Since at least two junctions at F_a are in $\tilde{C}$ for every $a \in F$, the set $\tilde{X}$ is a vertex cover of H.

The idea behind the reduction is as follows: Consider an edge $a = \{x, y\} \in F$. We interpret x_a^1 being expelled to mean that x is in the vertex cover. (In this case, the junctions of x are in the cycle cover.) Analogously, y is in the vertex cover if y_a^1 is expelled. The vertex z_a^1 is only absorbed if both x and y are in the vertex cover. If only one of x and y is in the vertex cover, z_a^1 forms a λ-cycle together with $z_a^2, \ldots, z_a^\lambda$.

We only considered G_1 when defining the terms "legally connected" and "legal." This is because in G_1, we lose weight one for putting x into the vertex cover since the junction $\{x_{a_\lambda}^1, x_{a_1}^1\}$ weighs zero. The other $\lambda - 1$ copies of the construction are only needed because z_a^1 must be part of some cycle if z_a^1 is not absorbed.

Lemma 2.3. *Let $\tilde{X}$ be a vertex cover of size $\tilde{n}$ of H. Then G contains an L-cycle cover $\tilde{C}$ with $w(\tilde{C}) = \sigma\lambda m - \tilde{n}$.*

Proof. We start by describing $\tilde{C}$ in G_1. For every vertex $x \in \tilde{X}$, the cycle consisting of all λ junctions is in $\tilde{C}$. Let $a = \{x, y\} \in F$ be any edge. Then either x or y or both are in $\tilde{X}$. If only x is in $\tilde{X}$, we let F_a absorb y_a^1 while z_a^1 is expelled. If only y is in $\tilde{X}$, we let F_a absorb x_a^1 while z_a^1 is again expelled. If both x and y are in $\tilde{X}$, then we let x_a^1 and y_a^1 be expelled while z_a^1 is absorbed.

We perform the same construction as for G_1 for all copies $G_2, \ldots, G_\lambda$. If z_a^1 is expelled, then $z_a^2, \ldots, z_a^\lambda$ are expelled as well. We let them form a λ-cycle in $\tilde{C}$.

Clearly, $\tilde{C}$ is legal. Furthermore, $\tilde{C}$ is an L-cycle cover: Every cycle either has a length of $\lambda \in L$ or lies totally inside a single L-gadget. All L-gadgets are healthy in $\tilde{C}$, thus $\tilde{C}$ is an L-cycle cover.

All edges of $\tilde{C}$ within $G_2, \ldots, G_\lambda$ have weight one. The only edges that connect different copies G_ξ and $G_{\xi'}$ are edges $\{z_a^\xi, z_a^{\xi+1}\}$ and $\{z_a^\lambda, x_a^1\}$, which have weight one as well. Almost all edges used in G_1 also have weight one; the only exception is one junction of weight zero for each $x \in \tilde{X}$. Since $|\tilde{X}| = \tilde{n}$, there are $\tilde{n}$ edges of weight zero in $\tilde{C}$. The graph G contains $\sigma\lambda m$ vertices, thus $\tilde{C}$ contains $\sigma\lambda m$ edges, $\sigma\lambda m - \tilde{n}$ of which have weight one. $\square$

Let C be an L-cycle cover of G and let $a \in F$. We define $W_{F_a}(C)$ as the sum of the weights of all internal edges of F_a plus half the number of z-edges in C at F_a. Analogously, $W_{G_\xi}(C)$ is the number of weight one edges with both vertices in G_ξ plus half the number of weight one edges with exactly one vertex in G_ξ.

Lemma 2.4. *Let C be an L-cycle cover and let j be the number of weight one junctions in C. Then $w(C) = j + \sum_{a \in F} W_{F_a}(C) + \sum_{\xi=2}^{\lambda} W_{G_\xi}(C)$.*

Proof. Every edge with both vertices in the same G_ξ is counted once. The only edges of weight one between different G_ξ are the edges $\{z_a^\xi, z_a^{\xi+1}\}$ and $\{z_a^\lambda, z_a^1\}$. These are counted with one half in both $W_{G_\xi}(C)$ and $W_{G_{\xi+1}}(C)$ for $2 \le \xi \le \lambda - 1$ or one half in both $W_{G_\xi}(C)$ and $W_{F_a}(C)$ for $\xi \in \{2, \lambda\}$. $\square$

In a legal L-cycle cover $\tilde{C}$ as described in Lemma 2.3, we have $W_{G_\xi}(\tilde{C}) = \sigma m$ for all $\xi \in \{2, \ldots, \lambda\}$ since every vertex in G_ξ is only incident to edges of weight one in $\tilde{C}$ by construction. Now we show that it is always best to traverse the gadgets legally and to keep the gadgets healthy.

Lemma 2.5. *Given an arbitrary L-cycle cover C, we can compute a legal L-cycle cover $\tilde{C}$ with $w(\tilde{C}) \ge w(C)$ in polynomial time.*

Proof. We proceed as follows to obtain $\tilde{C}$:

1. Let C' be C with all illegal edges removed.
2. For all $x \in X$ in arbitrary order: If at least one junction of x is in C, then put all junctions of x into C'.
3. For all $a = \{x, y\} \in F$ in arbitrary order: If neither the junctions of x nor the junctions of y are in C', choose arbitrarily one vertex of a, say x, and add all junctions of x to C'.
4. Rearrange C' within G_1 such that all clamps are healthy in C'.
5. Rearrange C' such that all $G_2, \ldots, G_\lambda$ are traversed exactly like G_1.
6. For all $a \in F$: If $z_a^1, \ldots, z_a^\xi$ are not absorbed, let them form a λ-cycle. Call the result $\tilde{C}$.

The running-time of the algorithm is polynomial. Moreover, $\tilde{C}$ is a legal L-cycle cover by construction. What remains is to prove $w(\tilde{C}) \ge w(C)$.

Let $w(C) = j + \sum_{a \in F} W_{F_a}(C) + \sum_{\xi=2}^{\lambda} W_{G_\xi}(C)$ be the weight of C according to Lemma 2.4, i. e., C contains j junctions of weight one. Analogously, let $w(\tilde{C}) = \tilde{\jmath} + \sum_{a \in F} W_{F_a}(\tilde{C}) + \sum_{\xi=2}^{\lambda} W_{G_\xi}(\tilde{C})$, i. e., $\tilde{\jmath}$ is the number of junctions of weight one in $\tilde{C}$.

All illegal edges have weight zero, and we do not remove any junctions. We have $W_{G_\xi}(\tilde{C}) = \sigma m$ for all ξ, which is maximal. Thus, no weight is lost in this way. What remains is to consider the internal edges of the gadgets and the z-edges.

Let $a = \{x, y\}$ be an arbitrary edge of H. If $W_{F_a}(C) \le W_{F_a}(\tilde{C})$, then nothing has to be shown. Those gadgets F_a with $W_{F_a}(C) > W_{F_a}(\tilde{C})$ remain to be considered. We have $W_{F_a}(\tilde{C}) \ge \sigma - 2$ and $W_{F_a}(C) \le \sigma - 1$ according to Lemma 2.2. Thus, $W_{F_a}(C) = \sigma - 1$

and $W_{F_a}(\tilde{C}) = \sigma - 2 = W_{F_a}(C) - 1$ for all $a \in F$ with $W_{F_a}(C) > W_{F_a}(\tilde{C})$. What remains to be proved is that for all such gadgets, there is a junction of weight one in $\tilde{C}$ that is not in C and can thus compensate for the loss of weight one in F_a. This means that we have to show that $\tilde{j}$ is at least j plus the number of edges a with $W_{F_a}(C) > W_{F_a}(\tilde{C})$.

If $W_{F_a}(C) = \sigma - 1$, then according to Lemma 2.2(4), the junctions at F_a in C (if there are any) belong to the same vertex. Since $W_{F_a}(\tilde{C}) = \sigma - 2$, all four junctions at F_a are in $\tilde{C}$. Thus, while executing the above algorithm, there is a moment at which at least one of, say, y's junctions at F_a is in C', and the junctions of x are added in the next step. We say that a vertex $\boldsymbol{x}$ **compensates** $\boldsymbol{F_a}$ if

1. $\tilde{C}$ contains x's junctions,
2. no junction of x at F_a is in C, and
3. at the moment at which x's junctions are added, C' already contains at least one junction of y at F_a.

Thus, every gadget F_a with $W_{F_a}(\tilde{C}) < W_{F_a}(C)$ is compensated by some vertex $x \in a$.

It remains to be shown that the number of gadgets that are compensated by some vertex is at most equal to the number of weight one junctions added to C'. Let $\eta \in \{0, \ldots, \lambda\}$ be the number of junctions of x in C. If $\eta = \lambda$, then x does not compensate any gadget. If $\eta = 0$, i. e., C does not contain any of x's junctions, then the junctions of x are added during Step 3 of the algorithm because there is some edge $a \in F$ with $x \in a$ such that there is no junction at all in C' at F_a before adding x's junctions. Thus, x does not compensate F_a. At most $\lambda - 1$ gadgets are compensated by x, and $\lambda - 1$ junctions of x have weight one. The case that remains is $\eta \in \{1, \ldots, \lambda - 1\}$. Then $\lambda - \eta$ junctions of x are added and at least $\lambda - \eta - 1$ of them have weight one. On the other hand, there are at least $\eta + 1$ gadgets F_a such that at least one junction of x at F_a is already in C: Every junction is at two gadgets, and thus η junctions are at $\eta + 1$ or more gadgets. Thus, at most $\lambda - \eta - 1$ gadgets are compensated by x. □

Finally, we prove the following counterpart to Lemma 2.3.

Lemma 2.6. *Let $\tilde{C}$ be the L-cycle cover constructed as described in the proof of Lemma 2.5 and let $\tilde{X} = \{x \mid x\text{'s junctions are in } \tilde{C}\}$ be the subset of X obtained from $\tilde{C}$. Choose $\tilde{n}$ such that $w(\tilde{C}) = \sigma\lambda m - \tilde{n}$. Then $|\tilde{X}| = \tilde{n}$.*

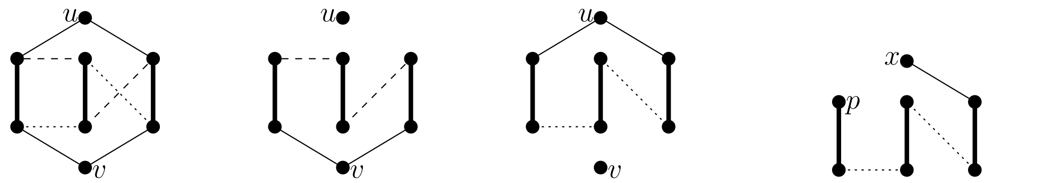
Proof. The proof is similar to the proof of Lemma 2.3. We set the weight of all junctions to one. With respect to the modified edge weights, the weight of $\tilde{C}$ is $\sigma\lambda m$. Thus, $\tilde{n}$ is the number of weight zero junctions in $\tilde{C}$, which is just $|\tilde{X}|$. □

Now we are prepared to prove the main theorem of this section.

Theorem 2.7. *For all $L \subseteq \mathcal{U}$ with $\overline{L} \not\subseteq \{3, 4\}$,* Max-$L$-UCC(0,1) *is* APX*-hard.*

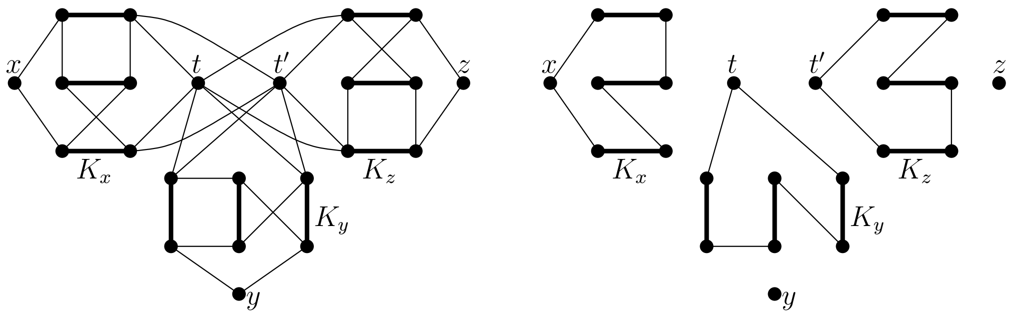
Proof. We show that the reduction presented is an L-reduction. Then the result follows from the APX-hardness of Min-Vertex-Cover(λ). Let $\mathrm{opt}(H)$ be the size of a minimum vertex cover of H and $\mathrm{opt}(G)$ be the weight of a maximum weight L-cycle cover of G. From Lemmas 2.3, 2.5, and 2.6, we obtain that $\mathrm{opt}(G) = \sigma\lambda m - \mathrm{opt}(H) \leq \sigma\lambda m$. Since H is λ-regular, we have $\mathrm{opt}(H) \geq n/2$. Thus,

$$\mathrm{opt}(G) \leq \sigma\lambda m = \sigma\lambda \cdot (\lambda n/2) \leq (\sigma\lambda^2) \cdot \mathrm{opt}(H).$$



(a) The clamp. (b) Absorbing v. (c) Absorbing u. (d) Illegal traversal of K_x.

Figure 4: A weighted L-clamp for $\{4\} \subseteq \overline{L} \subseteq \{3,4\}$ and how to traverse it. Bold edges have weight two; solid, dashed, and dotted edges have weight one.



(a) The weighted L-gadget. (b) How to absorb x.

Figure 5: A weighted L-gadget and how to use it.

Let C be an arbitrary L-cycle cover of G, $\tilde{C}$ be a legal L-cycle cover obtained from C as in Lemma 2.5, and $\tilde{X} \subseteq X$ obtained from $\tilde{C}$. Then

$$\big||\tilde{X}| - \mathrm{opt}(H)\big| = \big|w(\tilde{C}) - \mathrm{opt}(G)\big| \leq \big|w(C) - \mathrm{opt}(G)\big|,$$

which completes the proof. □

2.3 Adaption of the Reduction to Max-*L*-UCC

To prove the APX-hardness of Max-L-UCC for $\overline{L} \not\subseteq \{3\}$, all we have to do is to deal with $\overline{L} = \{4\}$ and $\overline{L} = \{3,4\}$. For all other sets L, the inapproximability follows from Theorem 2.7. We will adapt the reduction presented in the previous section.

To do this, we have to find an edge weighted analog of an L-clamp. We do not explicitly define the properties a weighted L-clamp has to fulfill. Instead, we just call the graph shown in Figure 4(a) a **weighted *L*-clamp** for $\overline{L} = \{3,4\}$ and $\overline{L} = \{4\}$.

The basic idea is that all three edges of weight two of the weighted clamp have to be traversed in a cycle cover. Since 4-cycles are forbidden, we have to take either the two dotted edges or the two dashed edges. Otherwise, we would have to take an edge of weight zero. Furthermore, if we take the dashed edges, we have to absorb v and to expel u, and if we take the dotted edges, we have to absorb u and to expel v (Figures 4(b) and 4(c)). Again, we would have to take edges of weight zero otherwise.

Using three weighted L-clamps K_x, K_y, K_z, we build an L-gadget as shown in Figure 5(a). Note that both t and t' can serve as a connector for each of the clamps. This weighted L-gadget has essentially the same properties as the L-gadgets of Section 2.1, which were stated as Lemma 2.2. The difference is that $\sigma = 32$ is no longer the number of vertices, but the number of vertices plus the number of edges of weight two.

Lemma 2.8. *Let G be an undirected graph with vertex set V and edge weights zero and* *one, and let $U \subseteq V$ be a weighted L-gadget with connectors x, y, z in G. Let C be an arbitrary L-cycle cover of G. Then the following properties hold:*

1. $w_U(C) \leq 31$.
2. *If there are 2α external edges at U in C, then $w_U(C) \leq 32 - \alpha$.*
3. *If U absorbs x, then there exists an L-cycle cover $\tilde{C}$ that differs from C only in the internal edges of U and has $w_U(\tilde{C}) = 30$. The same holds if U absorbs y or z.*
4. *Assume that there are two external edges at U in C that are incident to two different connectors. Then $w_U(C) \leq 30$.*

Proof. The only way to achieve $w_U(C) > 31$ is $w_U(C) = 32$, which requires that we have 23 internal edges including all nine edges of weight two. Since 4-cycles are forbidden, such an L-cycle cover does not exist.

If we have 2α external edges, then we have $23 - \alpha$ internal edges. At most nine of them are of weight two.

If U absorbs x, then we can achieve a weight of 30 by letting K_y and K_z absorb t_1 and t_2, respectively (Figure 5(b)). (We can also connect K_y and K_z via t and t' to obtain a 14-cycle. The weight would be the same.) In the same way, we can achieve weight 30 if U absorbs y or z.

The fourth claim remains to be proved. We have $w_U(C) \leq 31$ and 22 internal edges. If $w_U(C) > 30$, then $w_U(C) = 31$, and C contains all nine edges of weight two and no internal edge of weight zero of U. By symmetry, it suffices to consider the case that x is incident to one external edge. Figure 4(d) shows which edges are mandatory in order to keep all three edges of weight two. Since the cycle that contains x must be continued at p, vertex p is incident to an edge of weight zero in C, which proves the claim. $\square$

Given these properties, we can plug the L-gadget into the reduction described in the previous section to obtain the APX-hardness of Max-L-UCC for $\overline{L} = \{4\}$ and $\overline{L} = \{3, 4\}$. Together with Theorem 2.7, we obtain the following result.

Theorem 2.9. Max-L-UCC *is* APX*-hard for all L with $\overline{L} \not\subseteq \{3\}$ even if the edge weights are restricted to be zero, one, or two.*

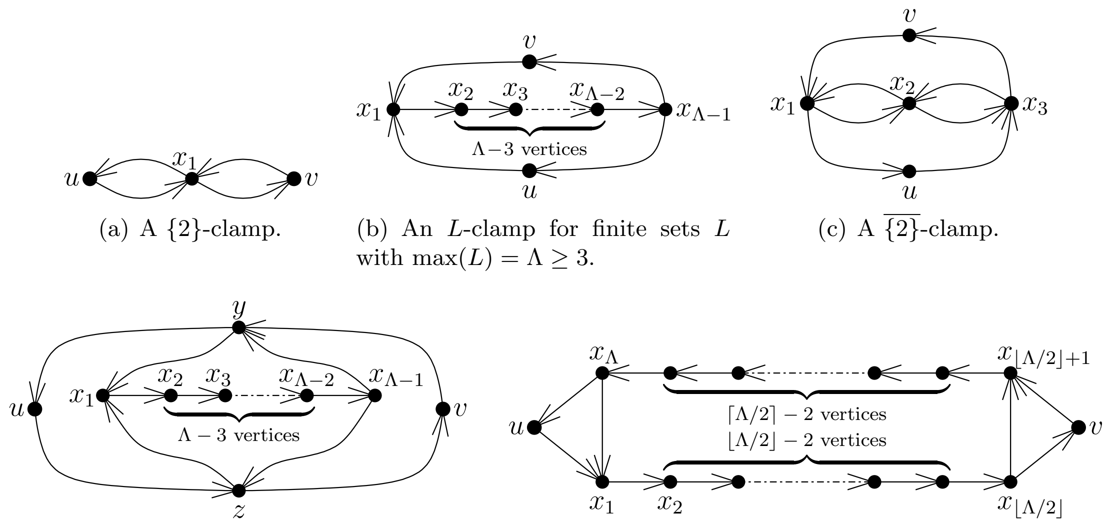
2.4 Clamps in Directed Graphs

The aim of this section is to prove a counterpart to Lemma 2.1 (for the existence of L-clamps) for directed graphs. Let $K = (V, E)$ be a directed graph and $u, v \in V$. Again, K_{-u}, K_{-v}, and K_{-u-v} denote the graphs obtained by deleting u, v, and both u and v, respectively. For $k \in \mathbb{N}$, K_u^k denotes the following graph: Let $y_1, \ldots, y_k \notin V$ be new vertices and add edges $(u, y_1), (y_1, y_2), \ldots, (y_k, v)$. For $k = 0$, we add the edge (u, v). The graph K_v^k is similarly defined, except that we now start at v, i. e., we add the edges $(v, y_1), (y_1, y_2), \ldots, (y_k, u)$. K_v^0 is K with the additional edge (v, u).

Now we can define clamps for directed graphs: Let $L \subseteq \mathcal{D}$. A directed graph $K = (V, E)$ with $u, v \in V$ is a **directed L-clamp** with connectors u and v if the following properties hold:

- Both K_{-u} and K_{-v} contain an L-cycle cover.
- Neither K nor K_{-u-v} nor K_u^k nor K_v^k for any $k \in \mathbb{N}$ contains an L-cycle cover.

Let us now prove that directed L-clamps exist for almost all L.



(a) A $\{2\}$-clamp.

(b) An L-clamp for finite sets L with $\max(L) = \Lambda \geq 3$.

(c) A $\overline{\{2\}}$-clamp.

(d) An L-clamp for $\Lambda \notin L$ and $\Lambda + 2 \in L$ with $\Lambda \geq 3$.

(e) An L-clamp for $\Lambda, \Lambda + 2 \notin L$ and $\Lambda + 1 \in L$ with $\Lambda \geq 4$.

Figure 6: Directed L-clamps. The connectors are u and v, the internal vertices are $x_1, x_2, \ldots$ and y, z.

Theorem 2.10. *Let $L \subseteq \mathcal{D}$ be non-empty. Then there exists a directed L-clamp if and only if $L \neq \mathcal{D}$.*

Proof. We first prove that directed L-clamps exist for all non-empty sets $L \subseteq \mathcal{D}$ with $L \neq \mathcal{D}$. We start by considering finite L. If L is finite, $\max(L) = \Lambda$ exists. For $L = \{2\}$, the graph shown in Figure 6(a) is a directed L-clamp: Either u or v forms a 2-cycle with x_1, and there are no other possibilities. Otherwise, we have $\Lambda \geq 3$. Figure 6(b) shows a directed L-clamp for this case, which is a directed variant of the undirected clamp shown in Figure 1(a).

Now we consider finite $\overline{L}$. Figure 6(c) shows an L-clamp for $\overline{L} = \{2\}$: x_1, x_2, and x_3 must be on the same path since length two is forbidden. This cycle must include u or v but cannot include both of them

Otherwise, $\max(\overline{L}) = \Lambda \geq 3$ and $\Lambda + 2 \in L$ and the graph shown in Figure 6(d) is an L-clamp: The vertices $x_1, \ldots, x_{\Lambda-1}$ must all be on the same cycle. Thus, either (y, x_1) or (z, x_1) is in the cycle cover. By symmetry, it suffices to consider the first case. Since $\Lambda \notin L$, the edge $(x_{\Lambda-1}, y)$ cannot be in the cycle cover. Thus, (v, y) and $(x_{\Lambda-1}, z)$ and hence (z, v) are in the cycle cover.

The case that remains to be considered is that both L and $\overline{L}$ are infinite. We distinguish two sub-cases. Either there exists a $\Lambda \geq 4$ with $\Lambda, \Lambda + 2 \notin L$ and $\Lambda + 1 \in L$. In this case, the graph shown in Figure 6(e) is an L-clamp: $x_1, \ldots, x_\Lambda$ must be on the same cycle. Since the lengths Λ and $\Lambda + 2$ are not allowed, either v or u is expelled and the other vertex is absorbed.

If no Λ exists with $\Lambda, \Lambda + 2 \notin L$ and $\Lambda + 1 \in L$ (but L and $\overline{L}$ are infinite), then there exists a $\Lambda \geq 3$ with $\Lambda \notin L$ and $\Lambda + 2 \in L$ and we can use the graph already used for finite $\overline{L}$ (Figure 6(d)) as a directed L-clamp.

Lemma 2.11 below shows that $\mathcal{D}$-clamps do not exist, which completes the proof. □

Lemma 2.11. *Let $G = (V, E)$ be a directed graph and let $u, v \in V$. If G_{-u} and G_{-v} both contain a cycle cover, then*

- *both G and G_{-u-v} contain cycle covers or*
- *all G_u^k and G_v^k for $k \in \mathbb{N}$ contain cycle covers.*

Proof. Let E_{-u} and E_{-v} be the sets of edges of the cycle covers of G_{-u} and G_{-v}, respectively. We construct two sequences of edges $P = (e_1, e_2, \ldots)$ and $P' = (e'_1, e'_2, \ldots)$. These sequences can be viewed as augmenting paths and we use them to construct cycle covers of G_{-u-v} and G or G_u^k and G_v^k. The sequence P is given uniquely by traversing edges of E_{-v} forwards and edges of E_{-u} backwards:

- $e_1 = (u, x_1)$ is the unique outgoing edge of $u = x_0$ in E_{-v}.
- If $e_i = (x_{i-1}, x_i) \in E_{-v}$, i. e., if i is odd, then $e_{i+1} = (x_{i+1}, x_i) \in E_{-u}$ is the unique incoming edge of x_i in E_{-u}.
- If $e_i = (x_i, x_{i-1}) \in E_{-u}$, i. e., if i is even, then $e_{i+1} = (x_i, x_{i+1}) \in E_{-v}$ is the unique outgoing edge of x_i in E_{-v}.
- If in any of the above steps no extension of P is possible, then stop.

Let $P = (e_1, \ldots, e_\ell)$. We observe two properties of the sequence P.

Lemma 2.12. *1. No edge appears more than once in P.*

2. If ℓ is odd, i. e., $e_\ell \in E_{-v}$, then $e_\ell = (x_{\ell-1}, u)$. If ℓ is even, i. e., $e_\ell \in E_{-u}$, then $e_\ell = (v, x_{\ell-1})$.

Proof. Assume the contrary of the first claim and let $e_i = e_j$ $(i \neq j)$ be an edge that appears at least twice in P such that i is minimal. If $i = 1$, then $e_j = (u, x_1) \in E_{-v}$. This would imply $e_{j-1} = (u, x_{j-2}) \in E_{-u}$, a contradiction. If $i > 1$, then assume $e_i = (x_{i-1}, x_i) \in E_{-v}$ without loss of generality. Since exactly one edge leaves x_{i-1} in E_{-u}, the edge $e_{i-1} = e_{j-1}$ is uniquely determined, which contradicts the assumption that i be minimal.

Let us now prove the second claim. Without loss of generality, we assume that the last edge e_ℓ belongs to E_{-v}. Let $e_\ell = (x_{\ell-1}, x_\ell)$. The path P cannot be extended, which implies that there does not exist an edge $(x_{\ell+1}, x_\ell) \in E_{-u}$. Since E_{-u} is a cycle cover of G_{-u}, this implies $x_\ell = u$ and completes the proof. $\square$

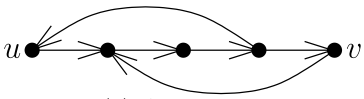
We build the sequence P' analogously, except that we start with the edge $e'_1 = (x'_1, v) \in E_{-u}$. Again, we traverse edges of E_{-v} forwards and edges of E_{-u} backwards. Let $P' = (e'_1, \ldots, e'_{\ell'})$.

No edge appears in both P and P' as can be proved similarly to the first claim of Lemma 2.12. Moreover, either P ends at u and P' ends at v or vice versa: We have $e_\ell = (x_{\ell-1}, u)$ if and only if $e'_{\ell'} = (v, x_{\ell'-1})$, and we have $e_\ell = (v, x_{\ell-1})$ if and only if $e'_{\ell'} = (x_{\ell'-1}, u)$. Let $P_{-u} \subseteq E_{-u}$ denote the set of edges of E_{-u} that are part of P. The sets P_{-v}, P'_{-u}, P'_{-v} are defined similarly.

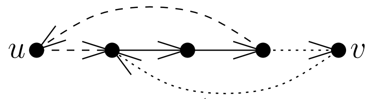
Two examples are shown in Figures 7 and 8: Figures 7(a) and 7(b) show a graph with its cycle covers, while Figure 7(c) depicts P and P', the former starting at u and ending at v and the latter starting at v and ending at u. Figures 8(a), 8(b), and 8(c) show another example graph, this time P starts and ends at u and P' starts and ends at v.

Our aim is now to construct cycle covers of G and G_{-u-v} or of G_u^k and G_v^k. We distinguish two cases. Let us start with the case that P starts at u and ends at v and, consequently, P' starts at v and ends at u. Then

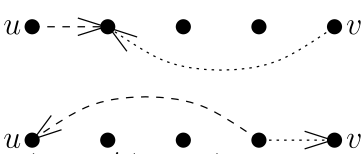
$$E_u^0 = (E_{-v} \setminus P_{-v}) \cup P_{-u} \cup \{(u, v)\}$$



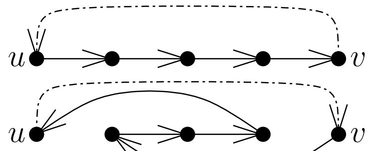
(a) A graph G.



(b) Cycle covers of G_{-v} (dashed and solid) and G_{-u} (dotted and solid).

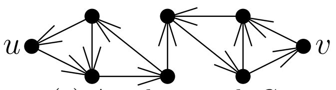


(c) P (top) and P' (bottom). Dashed and dotted edges belong to the cycle covers of G_{-v} and G_{-u}, respectively.

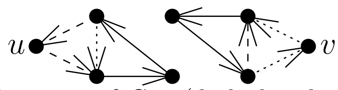


(d) Cycle covers of G_v^0 (top) and G_u^0 (bottom).

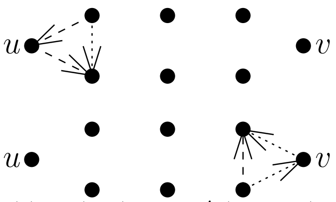
Figure 7: Constructing cycle covers of G_v^0 and G_u^0 from the sequences P and P'.



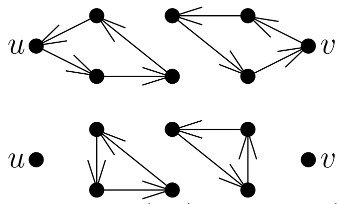
(a) Another graph G.



(b) Cycle covers of G_{-v} (dashed and solid) and G_{-u} (dotted and solid).



(c) P (top) and P' (bottom).



(d) Cycle covers of G (top) and G_{-u-v} (bottom).

Figure 8: Constructing cycle covers of G and G_{-u-v} from the sequences P and P'.

is a cycle cover of G_u^0. To prove this, we have to show $\mathrm{indeg}_{E_u^0}(x) = \mathrm{outdeg}_{E_u^0}(x) = 1$ for all $x \in V$:

- We removed the outgoing edge of u in E_{-v}, which is in P_{-v}. The incoming edge of u in E_{-v} is left. P_{-u} does not contain any edge incident to u and (u, v) is an outgoing edge of u. Thus, $\mathrm{indeg}_{E_u^0}(u) = \mathrm{outdeg}_{E_u^0}(u) = 1$.
- There is no edge incident to v in E_{-v}. P_{-u} contains an outgoing edge of v and (u, v) is an incoming edge of v. Thus, $\mathrm{indeg}_{E_u^0}(v) = \mathrm{outdeg}_{E_u^0}(v) = 1$.
- For all $x \in V \setminus \{u, v\}$, either both P_{-v} and P_{-u} contain an incoming edge of x or none of them does. Analogously, either both P_{-v} and P_{-u} contain an outgoing edge of x or none of them does. Thus, replacing P_{-v} by P_{-u} changes neither $\mathrm{indeg}(x)$ nor $\mathrm{outdeg}(x)$.

By replacing the edge (u, v) by a path $(u, y_1), \ldots, (y_k, v)$, we obtain a cycle cover of G_u^k for all $k \in \mathbb{N}$. A cycle cover of G_v^0 is obtained similarly:

$$E_v^0 = (E_{-u} \setminus P_{-u}) \cup P_{-v} \cup \{(v, u)\}.$$

As above, we get cycle covers of G_v^k by replacing (v, u) by a path $(v, y_1), \ldots, (y_k, u)$. Figure 7(d) shows an example how the new cycle covers are obtained.

The second case is that P starts and ends at u and P' starts and ends at v. Then

$$(E_{-v} \setminus P_{-u}) \cup P_{-v} \text{ and } (E_{-u} \setminus P'_{-v}) \cup P'_{-u}$$

are cycle covers of G and

$$(E_{-v} \setminus P_{-v}) \cup P_{-u} \text{ and } (E_{-u} \setminus P'_{-u}) \cup P'_{-v}$$

are cycle covers of G_{-u-v}. The proof is similar to the first case. Figure 8(d) shows an example. □

2.5 Intractability for Directed Graphs

From the hardness results in the previous sections and the work by Hell et al. [22], we obtain the NP-hardness and APX-hardness of L-DCC and Max-L-DCC(0,1), respectively, for all L with $2 \notin L$ and $\overline{L} \not\subseteq \{2,3,4\}$: We use the same reduction as for undirected cycle covers and replace every undirected edge $\{u,v\}$ by a pair of directed edges (u,v) and (v,u). However, this does not work if $2 \in L$ and also leaves open the cases when $\overline{L} \subsetneq \{2,3,4\}$. $\mathcal{D}$-DCC, Max-$\mathcal{D}$-DCC(0,1), and Max-$\mathcal{D}$-DCC can be solved in polynomial time, but the case $L = \{2\}$ is also easy: Replace two opposite edges (u,v) and (v,u) by an edge $\{u,v\}$ of weight $w(u,v) + w(v,u)$ and compute a matching of maximum weight on the undirected graph thus obtained.

We will settle the complexity of the directed cycle cover problems by showing that $L = \{2\}$ and $L = \mathcal{D}$ are the only tractable cases. For all other L, L-DCC is NP-hard and Max-L-DCC(0,1) and Max-L-DCC are APX-hard. Let us start by proving the APX-hardness.

Theorem 2.13. *Let $L \subseteq \mathcal{D}$ be a non-empty set. If $L \notin \{\{2\}, \mathcal{D}\}$, then* Max-$L$-DCC(0,1) *is* APX-*hard.*

Proof. We adapt the proof presented in Section 2.2. Since $L \neq \{2\}$, there exists a $\lambda \in L$ with $\lambda \geq 3$. Thus, Min-Vertex-Cover(λ) is APX-complete. All we need is such a λ and a directed L-clamp. Then we can reduce Min-Vertex-Cover(λ) to Max-L-DCC(0,1).

We use the L-clamps to build L-gadgets, which again should have the property that they absorb one of their connectors and expel the other two. In case of L being finite, the graph shown in Figure 9(a) is a directed L-gadget. In case of infinite L, we can build directed triple L-clamps exactly as for undirected graphs. Using these, we can build directed L-gadgets, which are simply directed variants of their undirected counterparts (Figure 9(b)).

The edge gadgets build the graph G_1: Let $x \in X$ be a vertex of H and $a_1, \ldots, a_\lambda \in F$ be the edges incident to x in H (in arbitrary order). Then we assign weight one to the edges $(x^1_{a_\xi}, x^1_{a_{\xi+1}})$ for all $\xi \in \{1, \ldots, \lambda - 1\}$. The edge $(x^1_{a_\lambda}, x^1_{a_1})$ has weight zero. These λ edges are called the junctions of x.

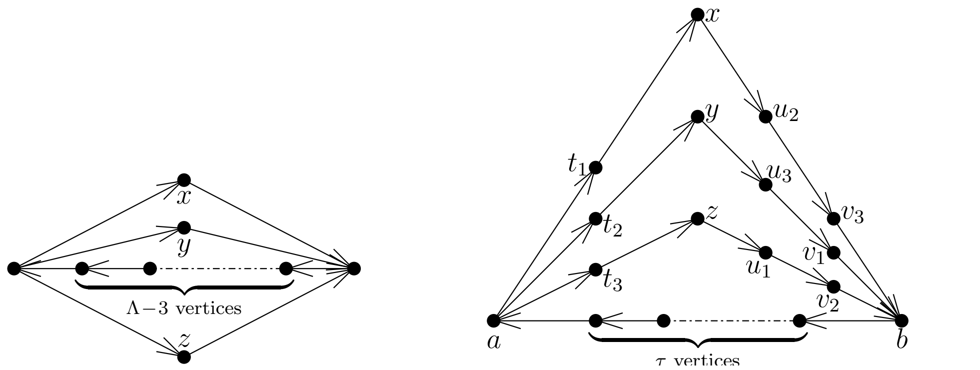
Again, $G_2, \ldots, G_\lambda$ are exact copies of G_1 except that weight one is assigned also to $(x^\xi_{a_\lambda}, x^\xi_{a_1})$ for all $\xi \in \{2, 3, \ldots, \lambda\}$.

Again, we let the z-vertices form λ-cycles: For all edges $a \in F$, we assign weight one to $(z^\xi_a, z^{\xi+1}_a)$ for $\xi \in \{1, 2, \ldots, \lambda - 1\}$ and to (z^λ_a, z^1_a).

Weight zero is assigned to all edges that are not mentioned.

The remainder of the proof goes along the same lines as the APX-hardness proof for undirected L-cycle covers. □

Note that the NP-hardness of L-DCC for $L \notin \{\{2\}, \mathcal{D}\}$ does not follow directly from the APX-hardness of Max-L-DCC(0,1): A famous counterexample is 2SAT, for which it is APX-hard to maximize the number of simultaneously satisfied clauses [27], although testing whether a 2CNF formula is satisfiable takes only linear time.



(a) L-gadget for finite L.

(b) L-gadget for infinite L with $\tau + 6 \in L$. The triple clamps are represented by their connectors t_i, u_i, v_i.

Figure 9: Directed L-gadgets with connectors x, y, z.

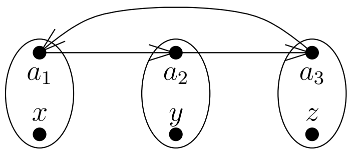


Figure 10: The construction for the NP-hardness of L-DCC from the viewpoint of $a = \{x, y, z\} \in F$. Each ellipse represents an L-clamp.

Theorem 2.14. *Let $L \subseteq \mathcal{D}$ be a non-empty set. If $L \notin \{\{2\}, \mathcal{D}\}$, then L-DCC is NP-hard.*

Proof. All we need is an L-clamp and some $\lambda \in L$ with $\lambda \geq 3$. We present a reduction from λ-XC (which is NP-complete since $\lambda \geq 3$) that is similar to the reduction of Hell et al. [22] used to prove the NP-hardness of L-UCC for $\overline{L} \not\subseteq \{3, 4\}$.

Let (X, F) be an instance of λ-XC. Note that we will construct a directed graph G as an instance of L-DCC, i. e., G is neither complete nor edge-weighted. For each $x \in X$, we have a vertex in G that we again call x. For $a = \{x_1, \ldots, x_\lambda\} \in F$, we construct a λ-cycle consisting of the vertices $a_1, \ldots, a_\lambda$. Then we add λ L-clamps $K_a^{x_\eta}$ with a_η and x_η as connectors for all $\eta \in \{1, \ldots, \lambda\}$. See Figure 10 for an example.

What remains to be shown is that G contains an L-cycle cover if and only if F is a "yes" instance of λ-XC. Assume first that there exists a subset $\tilde{F} \subseteq F$ such that $\bigcup_{a \in \tilde{F}} a = X$ and every element $x \in X$ is contained in exactly one set of $\tilde{F}$. We construct an L-cycle cover of G in which all clamps are healthy: Let $a = \{x_1, \ldots, x_\lambda\} \in F$. If $a \in \tilde{F}$, then let $K_a^{x_\eta}$ expel a_η and absorb x_η for all $\eta \in \{1, \ldots, \lambda\}$, and let $a_1, a_2, \ldots, a_\lambda$ form a λ-cycle. If $a \notin \tilde{F}$, let $K_a^{x_\eta}$ expel x_η and absorb a_η for all $\eta \in \{1, \ldots, \lambda\}$. All connectors are absorbed by exactly one clamp or are covered by a λ-cycle since $\tilde{F}$ is an exact cover.

Now we prove the reverse direction. Let C be an L-cycle cover of G. Then every clamp of G is healthy in C, i. e., it absorbs one of its connectors and expels the other one. Let $a = \{x_1, \ldots, x_\lambda\} \in F$ and assume that $K_a^{x_\eta}$ expels a_η. Since a_η must be part of a cycle in C, $(a_{\eta-1}, a_\eta)$ and $(a_\eta, a_{\eta+1})$ must be in C. We obtain that either all $a_1, \ldots, a_\lambda$ are absorbed by $K_a^{x_1}, \ldots, K_a^{x_\lambda}$ or that all are expelled by $K_a^{x_1}, \ldots, K_a^{x_\lambda}$. Now consider any $x \in X$ and let $a_1, a_2, \ldots, a_\ell \in F$ be all the sets that contain x. All clamps $K_{a_1}^x, \ldots, K_{a_\ell}^x$ are healthy, C is an L-cycle cover of G, and x is not incident to any further edges. Hence, there must be a unique a_i such that $K_{a_i}^x$ absorbs x. Thus,

$$\tilde{F} = \big\{a = \{x_1, \ldots, x_\lambda\} \in F \mid K_a^{x_\eta} \text{ absorbs } x_\eta \text{ for all } \eta \in \{1, \ldots, \lambda\}\big\}$$

is an exact cover of (X, F). $\square$

If the language $\{1^\lambda \mid \lambda \in L\}$ is in NP, then L-DCC is also in NP and therefore NP-complete if $L \notin \{\{2\}, \mathcal{D}\}$: We can nondeterministically guess a cycle cover and then check if $\lambda \in L$ for every cycle length λ occurring in that cover. Conversely, if $\{1^\lambda \mid \lambda \in L\}$ is not in NP, then L-DCC is not in NP either since there is a reduction of $\{1^\lambda \mid \lambda \in L\}$ to L-DCC: On input $x = 1^\lambda$, construct a graph G on λ vertices that consists solely of a Hamiltonian cycle. Then $x \in L$ if and only if G contains an L-cycle cover.

3 Approximation Algorithms

The goal of this section is to devise approximation algorithms for Max-L-UCC and Max-L-DCC that work for arbitrary L. The catch is that we have an uncountable number of problems Max-L-UCC and Max-L-DCC and for most L it is impossible to decide whether some cycle length is in L or not.

Assume, for instance, that we have an algorithm that solves Max-L-UCC for some set L that is not recursively enumerable. We enumerate all instances of Max-L-UCC and run the algorithm on these instances. This yields an enumeration of a subset of L. Since L is not recursively enumerable, there exist $\lambda \in L$ such that the algorithm never outputs λ-cycles. Now consider a graph with λ vertices where all edges have weight zero except for a Hamiltonian cycle of weight one edges. Then the Hamiltonian cycle is the unique optimum solution, but our algorithm does not output the λ-cycle, contradicting the assumption it solves Max-L-UCC.

One possibility to circumvent this problem would be to restrict ourselves to sets L such that $\{1^\lambda \mid \lambda \in L\}$ is in P. Another possibility to cope with this problem is to include the permitted cycle lengths in the input. However, while such restrictions are necessary for finding optimum solutions, it turns out that they are unnecessary for designing approximation algorithms.

A necessary and sufficient condition for a complete graph with n vertices to have an L-cycle cover is that there exist (not necessarily distinct) lengths $\lambda_1, \ldots, \lambda_k \in L$ for some $k \in \mathbb{N}$ with $\sum_{i=1}^{k} \lambda_i = n$. We call such an n ***L*-admissible** and define $\langle L \rangle = \{n \mid n \text{ is } L\text{-admissible}\}$. Although L can be arbitrarily complicated, $\langle L \rangle$ always allows efficient membership testing.

Lemma 3.1. *For all $L \subseteq \mathbb{N}$, there exists a finite set $L' \subseteq L$ with $\langle L' \rangle = \langle L \rangle$.*

Proof. Let $L_{\leq \ell} = \{n \in L \mid n \leq \ell\} \subseteq L$. Let $g_L \in \mathbb{N}$ be the greatest common divisor of all numbers in L. There exists an $\ell_0 \in L$ such that g_L is also the greatest common divisor of $L_{\leq \ell_0}$.

If $g_L \in L$, then $\langle \{g_L\} \rangle = \langle L \rangle$, and we are done. Thus, we assume $g_L \notin L$. There exist $\xi_1, \ldots, \xi_k \in \mathbb{Z}$ and $\lambda_1, \ldots, \lambda_k \in L_{\leq \ell_0}$ for some $k \in \mathbb{N}$ with $\sum_{i=1}^{k} \xi_i \lambda_i = g_L$. Let $\xi = \min_{1 \leq i \leq k} \xi_i$. We have $\xi < 0$ since $g_L \notin L$. Choose any $\lambda \in L_{\leq \ell_0}$ and let $\ell = -\xi\lambda \cdot \sum_{i=1}^{k} \lambda_i$. Let $n \in \langle L \rangle$ with $n \geq \ell$, let $m = \mathrm{mod}(n - \ell, \lambda)$, and let $s = \lfloor \frac{n-\ell}{\lambda} \rfloor$. We can write n as

$$n = \lambda s + m + \ell = \lambda s + \frac{m}{g_L} \cdot \sum_{i=1}^{k} \xi_i \lambda_i - \lambda\xi \cdot \sum_{i=1}^{k} \lambda_i = \lambda s + \sum_{i=1}^{k} (m\xi_i - \lambda\xi) \cdot \lambda_i.$$

Since $m < \lambda$ and $\xi_i \geq \xi < 0$, we have $(m\xi_i - \lambda\xi) \geq 0$ for all i. Hence, $\langle L_{\leq \ell_0} \rangle$ contains all elements $n \in \langle L \rangle$ with $n \geq \ell$. Elements of $\langle L \rangle$ smaller than ℓ are contained in $\langle L_{\leq \ell} \rangle \supseteq \langle L_{\leq \ell_0} \rangle$. Hence, $\langle L_{\leq \ell} \rangle = \langle L \rangle$ and $L' = L_{\leq \ell}$ is the finite set we are looking for. $\square$

ℓ	0	1	2	3	4	5
α_ℓ	0	1	1	0	0	1
β_ℓ	0	0	0	1	1	1

Table 1: A cycle cover on $n = 6k + \ell$ vertices will be decomposed into $k + \alpha_\ell$ singles and $k + \beta_\ell$ doubles.

For every fixed L, we can not only test in time polynomial in n whether n is L-admissible, but we can, provided that $n \in \langle L \rangle$, also find numbers $\lambda_1, \ldots, \lambda_k \in L'$ that add up to n, where $L' \subseteq L$ denotes a finite set with $\langle L \rangle = \langle L' \rangle$. This can be done via dynamic programming in time $O(n \cdot |L'|)$, which is $O(n)$ for fixed L.

Although $\langle L \rangle = \langle L' \rangle$, there are clearly graphs for which the weights of an optimal L-cycle cover and an optimal L'-cycle cover differ: Let $\lambda \in L \setminus L'$ and consider a λ-vertex graph where all edge weights are zero except for one Hamiltonian cycle of weight one edges. However, this does not matter for our approximation algorithms.

The two approximation algorithms presented in Sections 3.2 and 3.3 are based on a decomposition technique for cycle covers presented in Section 3.1.

3.1 Decomposing Cycle Covers

In this section, we present a decomposition technique for cycle covers. The technique can be applied to cycle covers of undirected graphs but also to directed cycle covers that do not contain 2-cycles.

A **single** is a single edge (or a path of length one) in a graph, while a **double** is a path of length two. Our aim is to decompose a cycle cover C on n vertices into roughly $n/6$ singles, $n/6$ doubles, and $n/6$ isolated vertices. If n is not divisible by six, we replace $n/6$ by $\lfloor n/6 \rfloor$ or $\lceil n/6 \rceil$: If $n = 6k + \ell$ for $k, \ell \in \mathbb{N}$ and $\ell \leq 5$, then we take $k + \alpha_\ell$ singles and $k + \beta_\ell$ doubles, where α_ℓ and β_ℓ are given in Table 1. Thus, we retain half of the edges of C. We aim to decompose the cycle covers such that at least half of the weight of the cycle cover is preserved.

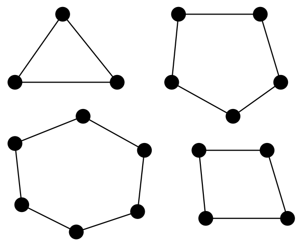
The reason why we decompose cycle covers into singles and doubles is the following: We cannot decompose them into longer paths in general since this does not work for $\{3\}$-cycle covers. If we restricted ourselves to decomposing the cycle covers into singles only, then 3-cycles would limit the weight preserved: We would retain only one third of the edges of the 3-cycles, thus at most one third of their weight in general. Finally, if we restricted ourselves to doubles, then 5-cycles would limit the weight we could obtain since we would retain only two of their five edges.

In our approximation algorithms, we exploit the following observation: If every cycle cover on n vertices can be decomposed into α singles and β doubles, then, for every L, every L-cycle cover on n vertices can be decomposed in the same way. This implies that we can build cycle covers from such a decomposition: Given α singles and β doubles, and $n - 2\alpha - 3\beta$ isolated vertices, we can join them to form an L-cycle cover. (The only restriction is that n must be L-admissible.)

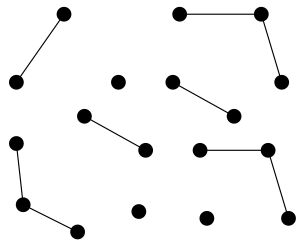
Let us now state the decomposition lemma.

Lemma 3.2. *Let $C = (V, E)$ be a cycle cover on $n = 6k + \ell$ vertices such that the length of each cycle is at least three. Let $w : E \to \mathbb{N}$ be an edge weight function.*

Then there exists a decomposition $D \subseteq E$ of C such that (V, D) consists of vertex-disjoint $k + \alpha_\ell$ singles, $k + \beta_\ell$ doubles, and $n - 5k - 3\beta_\ell - 2\alpha_\ell$ isolated vertices and $w(D) \geq w(E)/2$, where α_ℓ and β_ℓ are given in Table 1.



(a) A cycle cover.



(b) A decomposition of the cycle cover.

Figure 11: An example of a decomposition according to Lemma 3.2.

The decomposition can be done in polynomial time.

Figure 11 illustrates how a cycle cover is decomposed into singles and doubles.
Let us first prove some helpful lemmas.

Lemma 3.3. *Let $\lambda, \alpha, \beta \in \mathbb{N}$ with $\alpha + 2\beta \geq \lambda/2$ and $2\alpha + 3\beta \leq \lambda$. Then every cycle c of length λ can be decomposed into α singles and β doubles such that the weight of the decomposition is at least $w(c)/2$.*

Proof. Every single involves two vertices of c while every double involves three vertices. Thus, $2\alpha + 3\beta \leq \lambda$ is a necessary condition for c being decomposable into α singles and β doubles. It is also a sufficient condition.

We assign an arbitrary orientation to c. Let $e_0, \ldots, e_{\lambda-1}$ be the consecutive edges of c, where e_0 is chosen uniformly at random among the edges of c. We take α singles $e_0, e_2, \ldots, e_{2\alpha-2}$ and β doubles $(e_{2\alpha}, e_{2\alpha+1}), (e_{2\alpha+3}, e_{2\alpha+4}), \ldots, (e_{2\alpha+3\beta-3}, e_{2\alpha+3\beta-2})$. Since $2\alpha + 3\beta \leq \lambda$, this is a feasible decomposition. The probability that any fixed edge of c is included in the decomposition is $\frac{\alpha+2\beta}{\lambda}$. Thus, the expected weight of the decomposition is $\frac{\alpha+2\beta}{\lambda} \cdot w(c) \geq w(c)/2$. □

Lemma 3.4. *Let $\lambda \in \mathbb{N}$. Suppose that every cycle c of length λ can be decomposed into α singles and β doubles of weight at least $w(c)/2$. Then every cycle c' of length $\lambda + 6$ can be decomposed into $\alpha + 1$ singles and $\beta + 1$ doubles of weight at least $w(c')/2$.*

Proof. We have $\alpha + 2\beta \geq \lambda/2$ and $2\alpha + 3\beta \leq \lambda$. Thus, $\alpha + 1 + 2(\beta + 1) \geq (\lambda + 6)/2$ and $2(\alpha + 1) + 3(\beta + 1) \leq \lambda + 6$. The lemma follows from Lemma 3.3. □

Lemma 3.4 also holds if we consider more than one cycle: Assume that every collection of k cycles of lengths $\lambda_1, \ldots, \lambda_k$ can be decomposed into α singles and β doubles such that the weight of the decomposition is at least half the weight of the cycles. Then k cycles of lengths $\lambda_1 + 6, \lambda_2, \ldots, \lambda_k$ can be decomposed into $\alpha + 1$ singles and $\beta + 1$ doubles such that also at least half of the weight of the cycles is preserved. Due to Lemma 3.4, we can restrict ourselves to cycles of length at most eight in the following. The reason for this is the following: If we know how to decompose cycles of length λ, then we also know how to decompose cycles of length $\lambda + 6, \lambda + 12, \ldots$ from Lemma 3.4.

We are now prepared to prove Lemma 3.2.

Proof of Lemma 3.2. We prove the lemma by induction on the number of cycles. As the induction basis, we consider a cycle cover consisting of either a single cycle or of two odd cycles. Due to Lemma 3.4, we can restrict ourselves to considering cycles of length at most eight. Tables 2(a) and 2(b) show how to decompose a single cycle and two odd cycles, respectively. We always perform the decomposition such that the weight preserved

length	$\boldsymbol{\ell}$	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$
3	3	0	1
4	4	0	1
5	5	1	1
6	0	1	1
7	1	2	1
8	2	2	1

(a) One cycle.

lengths	$\boldsymbol{\ell}$	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	**decomposition**
3 + 3	0	1	1	$3 \rightsquigarrow (1,0) + 3 \rightsquigarrow (0,1)$
3 + 5	2	2	1	$3 \rightsquigarrow (1,0) + 5 \rightsquigarrow (1,1)$
				or $3 \rightsquigarrow (0,1) + 5 \rightsquigarrow (2,0)$
3 + 7	4	1	2	$3 \rightsquigarrow (1,0) + 7 \rightsquigarrow (0,2)$
				or $3 \rightsquigarrow (0,1) + 7 \rightsquigarrow (1,1)$
5 + 5	4	1	2	$5 \rightsquigarrow (0,1) + 5 \rightsquigarrow (1,1)$
5 + 7	0	2	2	$5 \rightsquigarrow (2,0) + 7 \rightsquigarrow (0,2)$
				or $5 \rightsquigarrow (1,1) + 7 \rightsquigarrow (1,1)$
7 + 7	2	3	2	$7 \rightsquigarrow (1,1) + 7 \rightsquigarrow (2,1)$

(b) Two odd cycles.

Table 2: The induction basis. The columns α and β show the number of singles and doubles needed, respectively. We denote by $\lambda \rightsquigarrow (\alpha, \beta)$ that a λ-cycle is decomposed into α singles and β doubles. If there are two lines for a case, then the option that yields more weight is chosen.

length	$\boldsymbol{\ell}$	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$
4	0,3,4,5	0	1
4	1,2	2	0
6	all	1	1
8	0,1,2,5	2	1
8	3,4	0	2

(a) Removing an even cycle.

lengths	$\boldsymbol{\ell}$	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	**decomposition**
3 + 3	all	1	1	$3 \rightsquigarrow (1,0) + 3 \rightsquigarrow (0,1)$
3 + 7	0,3,4,5	1	2	$3 \rightsquigarrow (1,0) + 7 \rightsquigarrow (0,2)$
				or $3 \rightsquigarrow (0,1) + 7 \rightsquigarrow (1,1)$
3 + 7	1,2	3	1	$3 \rightsquigarrow (1,0) + 7 \rightsquigarrow (2,1)$
				or $3 \rightsquigarrow (0,1) + 7 \rightsquigarrow (3,0)$
5 + 5	0,3,4,5	1	2	$5 \rightsquigarrow (0,1) + 5 \rightsquigarrow (1,1)$
5 + 5	1,2	3	1	$5 \rightsquigarrow (2,0) + 5 \rightsquigarrow (1,1)$
5 + 7	all	2	2	$5 \rightsquigarrow (2,0) + 7 \rightsquigarrow (0,2)$
				or $5 \rightsquigarrow (1,1) + 7 \rightsquigarrow (1,1)$
7 + 7	0,1,2,5	3	2	$7 \rightsquigarrow (1,1) + 7 \rightsquigarrow (2,1)$
7 + 7	3,4	1	3	$7 \rightsquigarrow (1,1) + 7 \rightsquigarrow (0,2)$

(b) Removing two odd cycles.

Table 3: Induction step.

is maximized. In particular, if there are two odd cycles of different length, we have two options in how to decompose these cycles, and we choose the one that yields the larger weight. Overall, we obtain a decomposition with an appropriate number of singles and doubles that preserves at least one half of the weight.

As the induction hypothesis, we assume that the lemma holds if the number of cycles is less than r. Assume that we have a cycle cover C consisting of r cycles. Let $n = 6k + \ell$ for the number of its vertices for $k, \ell \in \mathbb{N}$ and $\ell \leq 5$. We remove either an even cycle or two odd cycles. In the following, let C' be the new cycle cover obtained by removing one or two cycles from C. A little more care is needed than in the induction basis: Consider for instance the case of removing a 4-cycle. If $\ell = 4$, then C has to be decomposed into k singles and $k + 1$ doubles, while we have to take k singles and k doubles from C'. Thus, the 4-cycle has to be decomposed into a double. But if $\ell = 1$, then we need $k + 1$ singles and k doubles from C and $k - 1$ singles and k doubles from C'. Thus, the 4-cycle has to be decomposed into two singles. Overall, the 4-cycle has to be decomposed into a double if $\ell \in \{0, 3, 4, 5\}$ and into two singles if $\ell \in \{1, 2\}$. Similar case distinctions hold for all other cases. How to remove one even or two odd cycles is shown in Tables 3(a) and 3(b), respectively.

To complete the proof, we have to deal with the case of a 3- and a 5-cycle, which is slightly more complicated and not covered by Table 3(b). We run into trouble if, for instance, $\ell = 3$. In this case, we have to take two doubles. If the 5-cycle is much heavier than the 3-cycle, then it is impossible to preserve half of the weight of the two cycles. But we can avoid this problem: As long as there is an even cycle, we decompose this one. After that, as long as there are at least three odd cycles, we can choose two of them such that we do not have a pair of one $(3+6\xi)$-cycle and one $(5+6\xi')$-cycle for some $\xi, \xi' \in \mathbb{N}$. The only situation in which it can happen that we cannot avoid decomposing a $(3+6\xi)$-cycle and a $(5+6\xi')$-cycle is when there are only two cycles left. In this case, we have $\ell = 2$, and we have treated this case already in the induction basis. □

If we consider directed graphs where 2-cycles can also occur, only one third of the weight can be preserved. This can be done by decomposing the cycle cover into a matching of cardinality $\lceil n/3 \rceil$. (Every λ-cycle for can be decomposed into a matching of size up to $\lfloor \lambda/2 \rfloor \geq \lceil \lambda/3 \rceil$. The bottleneck are 3-cycles, which yield only one edge.)

An obvious question is whether the decomposition lemma can be improved in order to preserve more than half of the weight or more than one third of the weight if we additionally allow 2-cycles. Unfortunately, this is not the case.

A generic decomposition lemma states the following: For every $n \in \mathbb{N}$, every k-cycle cover (for $k \in \{2, 3\}$) on n vertices can be decomposed into α singles and β doubles such that at least a fraction r of the weight of the cycle cover is preserved. (As already mentioned, longer paths are impossible due to 3-cycles.) Lemma 3.2 instantiates this generic lemma with $\alpha \approx n/6$, $\beta \approx n/6$, and $r = 1/2$. In case of the presence of 2-cycles, we have sketched a decomposition with $\alpha \approx n/3$, $\beta = 0$, and $r = 1/3$.

Lemma 3.5. *No decomposition technique for* 3*-cycle covers can in general preserve more than one half of the weight of the* 3*-cycle covers.*

Furthermore, no decomposition technique for 2*-cycle covers can in general preserve more than one third of the weight of the* 2*-cycle covers.*

Proof. We exploit the fact that the fraction of edges that are preserved in a cycle cover decomposition is a lower bound for the fraction of the weight that can be preserved.

Since, in particular, $\{3\}$-cycle covers have to be decomposed, we cannot decompose the cycle cover into paths of length more than two. Now consider decomposing a $\{4\}$-cycle cover. Since paths of length 3 are not allowed, we have to discard two edges of every 4-cycle. Thus, at most 2 edges of every 4-cycle are preserved, which proves the first part of the lemma.

The second part follows analogously by considering 3-cycles and observing that paths of length two or more are not allowed. □

Overall, Lemma 3.5 shows that every approximation algorithm for Max-L-UCC or Max-L-DCC that works for arbitrary sets L and is purely decomposition-based achieves approximation ratios of at best 2 or 3, respectively. We achieve an approximation ratio of $8/3 < 3$ for Max-L-DCC by paying special attention to 2-cycles (Section 3.3).

3.2 Undirected Cycle Covers

Our approximation algorithm for Max-L-UCC (Algorithm 1) directly exploits Lemma 3.2.

Theorem 3.6. *Algorithm 1 is a factor* 2 *approximation algorithm for* Max-L-UCC *for all $L \subseteq \mathcal{U}$. Its running-time is $O(n^3)$.*

Proof. If L is infinite, we replace L by a finite set $L' \subseteq L$ with $\langle L' \rangle = \langle L \rangle$ according to Lemma 3.1. Algorithm 1 returns $\perp$ if and only if $n \notin \langle L \rangle$. Otherwise, an L-cycle

Input: undirected complete graph $G = (V, E)$, $|V| = n$; edge weights $w : E \to \mathbb{N}$
Output: an L-cycle cover C^{apx} of G if n is L-admissible, $\perp$ otherwise

if $n \notin \langle L \rangle$ **then**
  return $\perp$
compute a cycle cover C^{init} in G of maximum weight
decompose C^{init} into a set $D \subseteq C^{\text{init}}$ of edges according to Lemma 3.2
join the singles and doubles in D to obtain an L-cycle cover C^{apx}
return C^{apx}

Algorithm 1: A 2-approximation algorithm for Max-L-UCC.

Input: directed complete graph $G = (V, E)$, $|V| = n$; edge weights $w : E \to \mathbb{N}$
Output: an L-cycle cover C^{apx} of G if n is L-admissible, $\perp$ otherwise

if $n \notin \langle L \rangle$ **then**
  return $\perp$
if $2 \in L$ and $3 \in L$ **then**
  compute a cycle cover C^{init} (without restrictions)
  for all even cycles c of C^{init} **do**
    take every other edge of c such that at least one half of c's weight is preserved
    add the converse edges to obtain 2-cycles; add these cycles to C^{apx}
  for all odd cycles c of C^{init} **do**
    take every other edge and one path of length two of c such that at least one half of c's weight is preserved
    add edges to obtain 2-cycles plus one 3-cycle; add these cycles to C^{apx}
else if $2 \in L$, $3 \notin L$ **then**
  compute a matching M of maximum weight of cardinality at most $D(n, L)$
  join the edges of M to form an L-cycle cover C^{apx}
else ($2 \notin L$)
  compute a 4/3-approximation C_3^{init} to an optimal 3-cycle cover
  decompose C_3^{init} into a set $D \subseteq C_3^{\text{init}}$ of edges according to Lemma 3.2
  join the singles and doubles in D to obtain an L-cycle C^{apx}
return C^{apx}

Algorithm 2: A factor 8/3 approximation algorithm for Max-L-DCC.

cover C^{apx} is returned. Let $C^\star$ denote an L-cycle cover of maximum weight of G. We have $w(C^\star) \leq w(C^{\text{init}}) \leq 2 \cdot w(D) \leq 2 \cdot w(C^{\text{apx}})$. The first inequality holds because L-cycle covers are special cases of cycle covers. The second inequality holds due to the decomposition lemma (Lemma 3.2). The last inequality holds since no weight is lost during the joining. Overall, the algorithm achieves an approximation ratio of 2.

The running-time of the algorithm is dominated by the time needed to compute the initial cycle cover, which is $O(n^3)$ [1, Chapter 12]. □

3.3 Directed Cycle Covers

In the following, let C^{opt} be an L-cycle cover of maximum weight. Let w_λ denote the weight of the λ-cycles in C^{opt}, i. e., $w(C^{\text{opt}}) = \sum_{\lambda \geq 2} w_\lambda$.

We distinguish three cases: First, $2 \notin L$, second, $2 \in L$ and $3 \notin L$, and third, $2, 3 \in L$.

We use the decomposition lemma (Lemma 3.2) only if $2 \notin L$. In this case, the weight of an optimal L-cycle cover is at most the weight of an optimal 3-cycle cover C_3^{opt}. Thus, we proceed as follows: First, we compute a 4/3 approximation C_3^{init} for Max-3-DCC, which can be done by using the algorithm of Bläser et al. [7]. We have $w(C_3^{\text{init}}) \geq \frac{3}{4} \cdot w(C_3^{\text{opt}}) \geq \frac{3}{4} \cdot w(C^{\text{opt}})$. Now we decompose C_3^{init} into a collection D of singles and doubles according to Lemma 3.2. Finally, we join the singles, doubles, and isolated vertices of D to form an L-cycle cover C^{apx}. We obtain a factor 8/3 approximation for the case that $2 \notin L$:

$$w(C^{\text{apx}}) \geq w(D) \geq \frac{1}{2} \cdot w(C_3^{\text{init}}) \geq \frac{3}{8} \cdot w(C^{\text{opt}}).$$

Now we consider the case that $2 \in L$ and $3 \notin L$. In this case, a matching-based algorithm achieves an approximation ratio of 5/2: We compute a matching of a certain cardinality, which we will specify in a moment, and then we join the edges of the matching to obtain an L-cycle cover. The cardinality of the matching is chosen such that an L-cycle cover can be built from such a matching. A λ-cycle yields a matching of cardinality $\lfloor \lambda/2 \rfloor$. Thus, a matching of cardinality d in a graph of n vertices can be extended to form an L-cycle cover if and only if $d \leq D(n, L)$, where

$$D(n, L) = \max \left\{ \sum_{i=1}^{k} \lfloor \lambda_i/2 \rfloor \mid k \in \mathbb{N}, \sum_{i=1}^{k} \lambda_i = n, \text{ and } \lambda_i \in L \text{ for } 1 \leq i \leq k \right\} \leq \frac{n}{2}.$$

Given L, we can compute $D(n, L)$ using dynamic programming. Let us now estimate the weight of a matching of cardinality at most $D(n, L)$ that has maximum weight among all such matchings. From C^{opt}, we obtain a matching with a weight of at least

$$\sum_{\lambda \geq 2} \frac{1}{\lambda} \cdot \left\lfloor \frac{\lambda}{2} \right\rfloor \cdot w_\lambda \geq \sum_{\lambda \geq 2} \frac{2}{5} \cdot w_\lambda = \frac{2}{5} \cdot w(C^{\text{opt}}).$$

The reason is that $w_3 = 0$ because $3 \notin L$ and that $\min_{\lambda \in \{2,4,5,6,7,\ldots\}} \frac{1}{\lambda} \cdot \lfloor \lambda/2 \rfloor \geq 2/5$. Thus, by computing a maximum-weight matching M of cardinality at most $D(n, L) \geq 2n/5$ and joining the edges to form an L-cycle cover C^{apx}, we obtain a factor 5/2 approximation.

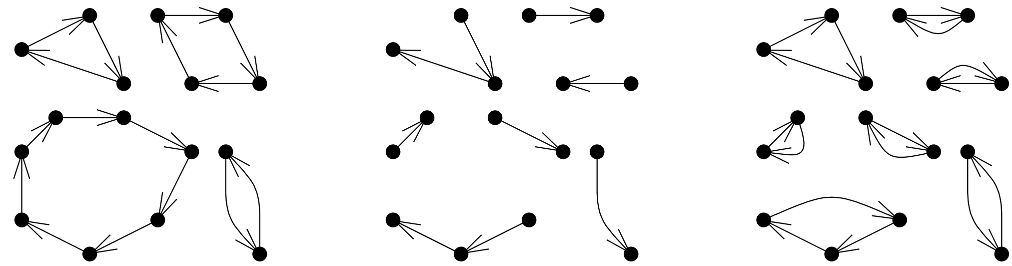
What remains to be considered is the case that $2, 3 \in L$. In this case, we start by computing an initial cycle cover C^{init} (without any restrictions). Then we do the following: For every even cycle, we take every other edge such that at least one half of its weight is preserved. For every edge thus obtained, we add the converse edge to obtain a collection of 2-cycles. For every odd cycle, we take every other edge and one path of length two such that at least half of the weight is preserved. Then we add edges to obtain 2-cycles and one 3-cycle. In this way, we obtain a $\{2, 3\}$-cycle cover C^{apx}, which is also an L-cycle cover. We have $w(C^{\text{apx}}) \geq \frac{1}{2} \cdot w(C^{\text{init}}) \geq \frac{1}{2} \cdot w(C^{\text{opt}})$. Figure 12 shows an example.

Our approximation algorithm is summarized as Algorithm 2. The running-time of the algorithm of Bläser et al. is polynomial [7] and all other steps can be executed in polynomial time as well. Thus, the running-time of Algorithm 2 is also polynomial.

Theorem 3.7. *Algorithm 2 is a factor* 8/3 *approximation algorithm for* Max-L-UCC *for all non-empty sets* $L \subseteq \mathcal{D}$. *Its running-time is polynomial.*

4 Conclusions

For almost all L, finding L-cycle covers is NP-hard and finding L-cycle covers of maximum weight is APX-hard. Table 4 shows an overview. Although this shows that computing restricted cycle covers is generally very hard, we have proved that L-cycle covers of maximum weight can be approximated within a constant factor in polynomial time for all L.



(a) Initial cycle cover C^{init}. (b) Decomposition of C^{init}. (c) $\{2,3\}$-cycle cover C^{apx}.

Figure 12: Sketch of the algorithm for $\{2,3\} \subseteq L$.

	$\boldsymbol{L}$-UCC	Max-$\boldsymbol{L}$-UCC(0,1)	Max-$\boldsymbol{L}$-UCC
$\overline{L} = \emptyset$	in P	in PO	in PO
$\overline{L} = \{3\}$	in P	in PO	
$\overline{L} = \{4\}, \{3,4\}$			APX-complete
$\overline{L} \not\subseteq \{3,4\}$	NP-hard	APX-hard	APX-hard

(a) Undirected cycle covers.

	$\boldsymbol{L}$-DCC	Max-$\boldsymbol{L}$-DCC(0,1)	Max-$\boldsymbol{L}$-DCC
$L = \{2\}, \mathcal{D}$	in P	in PO	in PO
$L \notin \{\{2\}, \mathcal{D}\}$	NP-hard	APX-hard	APX-hard

(b) Directed cycle covers.

Table 4: The complexity of computing L-cycle covers.

For directed graphs, we have settled the complexity: If $L = \{2\}$ or $L = \mathcal{D}$, then L-DCC, Max-L-DCC(0,1), and Max-L-DCC are solvable in polynomial time, otherwise they are intractable. For undirected graphs, the status of only five cycle cover problems remains open: L-UCC and Max-L-UCC(0,1) for $\overline{L} = \{4\}, \{3,4\}$ and Max-4-UCC.

There are some reasons for optimism that L-UCC and Max-L-UCC(0,1) for $\overline{L} = \{4\}, \{3,4\}$ are solvable in polynomial time: Hartvigsen [18] devised a polynomial-time algorithm for finding $\overline{\{4\}}$-cycle covers in bipartite graphs (forbidding 3-cycles does not change the problem for bipartite graphs). Moreover, there are augmenting path theorems for L-cycle covers for all L with $\overline{L} \subseteq \{3,4\}$ [28], which includes the two cases that are known to be polynomial-time solvable. Augmenting path theorems are often a building block for matching algorithms. But there are also augmenting path theorems for $L \subseteq \{3,4\}$ [28], even though these L-cycle cover problems are intractable.

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