Chapter 1

Traffic Flows

Consider a directed graph (V, E) with given O-D pairs (r_i, s_i). Each O-D pair has an associated *demand* $d_i \geq 0$ of flow from r_i to s_i. In the context of traffic flows – as well as in many other situations – it is natural to investigate flows from r_i to s_i that decompose into simple $r_i - s_i$ path flows. Thus let us define a *traffic flow* from r_i to s_i (of value d_i) to be a vector $\mathbf{x}^{(i)} \in \mathbb{R}_+^E$ which can be written as

$$\mathbf{x}^{(i)} = \sum_{P \in \mathcal{P}_i} \lambda_P P \qquad \text{with } \lambda_P \geq 0,\ \sum_P \lambda_P = d_i, \tag{1.1}$$

where $\mathcal{P}_i$ is the set of (incidence vectors of) simple directed $r_i - s_i$ paths.

Given traffic flows $\mathbf{x}^{(i)} \in \mathbb{R}_+^E$ for all O-D pairs, we obtain a corresponding (*total*) *flow*

$$\mathbf{x} = \sum_i \mathbf{x}^{(i)} \in \mathbb{R}_+^E.$$

We denote the set of traffic flows of value d_i from r_i to s_i by $X_{d_i}^{(i)} \subseteq \mathbb{R}^E$ and the set of total flows (relative to a given demand vector $\mathbf{d} = (d_i)$) by

$$X_{\mathbf{d}} = \sum_i X_{d_i}^{(i)}.$$

REMARK. The restriction to traffic flows as opposed to general (non-negative) flows is partly due to tradition and, in any case, not very essential. Recall that, in general, a non-negative flow $\mathbf{x}^{(i)}$ from r_i to s_i can be decomposed into directed paths and circuits in the form

$$\mathbf{x}^{(i)} = \sum_{P \in \mathcal{P}_i} \lambda_P P + \sum_{C \in \mathcal{C}} \lambda_C C \qquad \text{with } \lambda_P, \lambda_C \geq 0,$$

where $\mathcal{C}$ is the set of (incidence vectors of) directed circuits in (V, E). The decomposition, however, is not unique. In particular, a traffic flow $\mathbf{x}^{(i)}$ as in (1.1) is not necessarily *acyclic* (*i.e.*, it may well happen that $\mathbf{x}^{(i)} > 0$ on a directed circuit $C \in \mathcal{C}$). The traffic flows we will be interested in, however, are "essentially acyclic" (*cf.* Corollary 1.1).

In the classical traffic flow model due to Wardrop (1952), it is assumed that the "travel time" (sometimes also referred to as "latency") along the edge $e \in E$ is an increasing function of the total flow $x_e = \sum_i x_e^{(i)}$. More precisely, let us assume that each edge $e \in E$ is endowed with a non-negative, continuous and non-decreasing *cost function*

$$c_e : \mathbb{R}_+ \to \mathbb{R}_+.$$

Correspondingly, we define the *cost* ("total travel time") of a total flow $\mathbf{x} \in X_\mathbf{d}$ as

$$C(\mathbf{x}) := \sum_e c_e(x_e)x_e.$$

A *min cost flow* is a total flow $\mathbf{x}^* \in X_\mathbf{d}$ that minimizes the cost $C(\mathbf{x})$ over all $\mathbf{x} \in X_\mathbf{d}$.

A somewhat different (though related) concept is that of a "Nash equilibrium flow", defined as follows. Let $\overline{\mathbf{x}} \in X_\mathbf{d}$ be a total flow. According to (1.1), $\overline{\mathbf{x}}$ has a path decomposition

$$\overline{\mathbf{x}} = \sum_i \sum_{P \in \mathcal{P}_i} \lambda_P^{(i)} P.$$

We say that $\overline{\mathbf{x}}$ is a *Nash equilibrium* or simply a *Nash flow* if each $P \in \mathcal{P}_i$ with $\lambda_P^{(i)} > 0$ is a min cost path from r_i to s_i relative to the constant edge costs $\overline{c}_e = c_e(\overline{x}_e)$. Intuitively, imagine that the corresponding traffic flows $\overline{\mathbf{x}}^{(i)}$ are created by a large number of individual travelers from r_i to s_i. An individual traveler, constituting an "infinitesimally small" fraction of the flow from r_i to s_i then experiences the edge costs (travel times) $\overline{c}_e = c_e(\overline{x}_e)$ caused by the total flow $\overline{\mathbf{x}}$ (of all other travelers). So he would then decide to switch from its current strategy ("follow P") to a cheaper one, unless his current path P is already cost minimal relative to the edge costs $\overline{c}_e$. In other words, Nash flows are exactly those total flows that are *stable* in the sense that no individual traveller ("player") has any incentive to switch from its current path ("strategy") to another one, given the decisions of the others.

At the first glance, it may seem that the property of being Nash depends on the path decomposition of $\overline{\mathbf{x}} \in X_\mathbf{d}$. However, this is not the case:

Theorem 1.1 *The total flow $\overline{\mathbf{x}} \in X_{\mathbf{d}}$ is Nash if and only if $\overline{\mathbf{x}}$ is cost minimal among all $\mathbf{x} \in X_d$ relative to the constant edge costs $\overline{c}_e = c_e(\overline{x}_e)$.*

Proof. Let $\overline{\mathbf{x}} \in X_{\mathbf{d}}$ and $\overline{c}_e = c_e(\overline{x}_e)$, $e \in E$. A cost minimal flow in X_d relative to the constant edge costs $\overline{c}_e$ would (due to the absence of capacity constraints) simply route all the demand d_i along min cost paths from r_i to s_i. Hence if $\overline{\mathbf{x}}$ is a min cost flow in $X_{\mathbf{d}}$ relative to the edge costs $\overline{c}_e$, then $\overline{\mathbf{x}}$ must be Nash. The converse follows in the same way. $\square$

In the modern literature, the Nash flow model is also referred to as *selfish routing model*. It may be used to model and analyze traffic flows on roads or electronic networks, assuming the absence of a "government" or "network authority" that would regulate the traffic by telling all individuals how they should travel between the various O-D pairs so as to form in total a min cost flow. As a consequence, the *Nash cost*, *i.e.*, the cost $C(\overline{x})$ of a Nash flow $\overline{\mathbf{x}} \in X_{\mathbf{d}}$ will in general exceed that of a min cost flow.

We illustrate the difference between min cost flows and Nash flows with some examples. The smallest possible nontrivial example is due to Pigou (1920): There is a single O-D pair (r, s) and two links, say, e and f joining r to s. The first link has constant edge cost $c_e(x) = 1$ and the second link has cost $c_f(x) = x$.

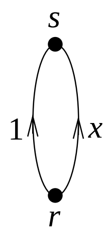


Figure 1.1: Pigou's network

Consider the demand $d = 1$. So any flow $\mathbf{x} \in X_d$ would route a certain amount $x_f = \lambda$, $\lambda \in [0, 1]$ on the second link and $x_e = 1 - \lambda$ on the first link, at cost

$$C(x) = 1 \cdot (1 - \lambda) + \lambda \cdot \lambda = 1 - \lambda(1 - \lambda).$$

Hence the corresponding min cost flow $\mathbf{x}^*$ is given by $x_e^* = x_f^* = \frac{1}{2}$ with cost $C(\mathbf{x}^*) = \frac{3}{4}$. The unique Nash flow $\overline{\mathbf{x}}$, on the other hand, is obtained by $\overline{x}_e = 0$ and $\overline{x}_f = 1$, with Nash cost $C(\overline{\mathbf{x}}) = 1$.

The cost of a Nash flow can even be arbitrarily bad, compared to the minimum cost: Replace the linear cost function $c(x) = x$ in Pigou's network by the "steeper" nonlinear function $c(x) = x^p$, for some $p \in \mathbb{N}$.

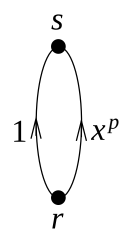


Figure 1.2: Pigou's network (non-linear variant)

The Nash flow $\bar{\mathbf{x}}$ will remain unchanged with cost $C(\bar{\mathbf{x}}) = 1$, whereas the min cost flow $\mathbf{x}^*$ has $x^*_f = \lambda$, $x^*_e = 1 - \lambda$, where $\lambda \in [0, 1]$ is a solution of

$$C(\mathbf{x}^*) = \min 1 \cdot (1 - \lambda) + \lambda^p \cdot \lambda = \min 1 - \lambda(1 - \lambda^p).$$

The solution is $\lambda = \sqrt[p]{1/(p+1)}$, so $C(x^*) = 1 - \sqrt[p]{1/(p+1)}\frac{p}{p+1}$ (which tends to zero as $p \to \infty$).

An interesting and (at the first glance) surprising phenomenon, stipulating again the different behavior of Nash flows and min cost flows, is described in the next example. Consider a network as in figure 1.3 below.

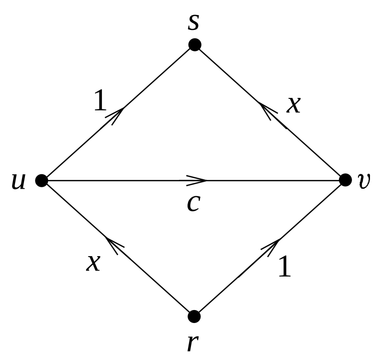


Figure 1.3: The Braess network

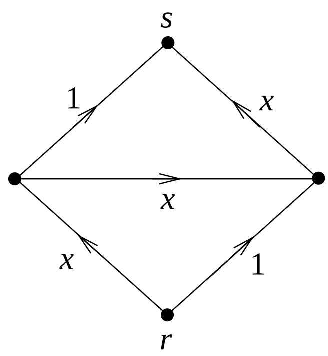
The demand from r to s is $d = 1$. We investigate how the Nash flow changes with the constant edge cost $c \in \mathbb{R}$ on the arc (u, v). For $c = \frac{1}{2}$, the Nash flow (*cf.* Ex. 1) $\bar{\mathbf{x}}$ would route half of the demand $d = 1$ along each of the paths $r - u - s$ resp. $r - v - s$. The corresponding Nash cost is $C(\bar{\mathbf{x}}) = 2 \cdot (\frac{1}{2} + \frac{1}{4}) = 3/2$.

For $c = 0$, however, the unique Nash flow would route all the demand along $r - u - v - s$, resulting in a Nash cost of $C(\bar{\mathbf{x}}) = 2$. So the example, due to Braess

(1968) shows that *decreasing* the edge costs may result in an *increase* of the Nash cost (*Braess' paradox*).

Ex. 1.1 *Compute Nash flows and min cost flows in the Braess network (with demand $d = 1$) for all $c \in \mathbb{R}_+$.*

Ex. 1.2 *Compute the Nash flow in the network below for each demand $d \geq 0$.*



Remark. As we have seen, Nash flows may be far from optimal in some cases. In this sense, the examples above (Pigou, Braess) disprove – at least in theory – the (neo–)liberal claim, stating that a free market would automatically "optimize itself". In practice, the discrepancy between Nash flows ("user optimized") and min cost flows ("system optimized") is often further enlarged by the fact that the network administration and the network users may have a different opinion about the edge costs $c_e(x)$. For example, in a road network, the individual car driver would probably seek to minimize its travel time, whereas the government might want to reduce the overall CO_2-emission.

1.1 Existence of Nash flows

Here we show that Nash flows always exist and are even unique in some sense (*cf.* Theorem 1.4 below). We first review the following basic fact from convex optimization.

Theorem 1.2 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable convex function. Then*

$$f(\mathbf{x}) \geq f(\overline{\mathbf{x}}) + \nabla f(\overline{\mathbf{x}})(\mathbf{x} - \overline{\mathbf{x}})$$

holds for any two points $\overline{\mathbf{x}}, \mathbf{x} \in \mathbb{R}^n$. In particular, if $X \subseteq \mathbb{R}^n$ is a convex set, then $\overline{\mathbf{x}} \in X$ is a minimizer of f on X if and only if

$$\nabla f(\overline{\mathbf{x}})(\mathbf{x} - \overline{\mathbf{x}}) \geq 0 \qquad \forall \mathbf{x} \in X.$$

Proof. Convexity of f implies for (small) $t \in [0, 1]$

$$\begin{aligned} tf(\mathbf{x}) + (1-t)f(\overline{\mathbf{x}}) &\geq f(\overline{\mathbf{x}} + t(\mathbf{x} - \overline{\mathbf{x}})) \\ &= f(\overline{\mathbf{x}}) + t\nabla f(\overline{\mathbf{x}})(\mathbf{x} - \overline{\mathbf{x}}) + o(t). \end{aligned}$$

Subtracting $f(\overline{\mathbf{x}})$ on both sides, we obtain

$$t(f(\mathbf{x}) - f(\overline{\mathbf{x}})) \geq t\nabla f(\overline{\mathbf{x}})(\mathbf{x} - \overline{\mathbf{x}}) + o(t)$$

and the claim follows by letting $t \to 0$. $\square$

There is a general phenomenon, known as *variational principle*, which is observed in many places in physical or social sciences, stating that equilibrium states of a given system can alternatively be characterized as those states that minimize a certain quantity like, *e.g.*, energy or tension. This principle also applies in our case: Consider the function $N : \mathbb{R}_+^E \to \mathbb{R}$ defined by

$$N(\mathbf{x}) = \sum_{e \in E} \int_0^{x_e} c(t)dt.$$

Remark. The value $N(\mathbf{x})$ may be interpreted as follows. Imagine the flow $\mathbf{x} \in X_{\mathbf{d}}$ being formed by a large number of travelers, each contributing a small amount $\Delta > 0$ to one of the path flows in $\mathbf{x}$. Assume that the travelers enter the network one after the other. When a particular traveler enters, there is a current flow $\mathbf{x}' \leq \mathbf{x}$ formed by the previous travelers. So he experiences a cost of

$$\Delta \sum_{e \in P} c_e(x'_e) \approx \sum_{e \in P} \int_{x'_e}^{x'_e + \Delta} c(t)dt$$

along his path P. In this sense, $N(\mathbf{x})$ equals the (infinite) sum of the travel costs experienced by the individuals, as they enter the network (in some order). In contrast to $C(\mathbf{x})$, the value $N(\mathbf{x})$ thus neglects the impact that a particular traveler has on the costs for the "previous" travelers (*cf.* also Ex. 1.6).

The function $N : \mathbb{R}_+^E \to \mathbb{R}$ is convex (*cf.* Ex. 1.3) and differentiable with gradient

$$\nabla N(\mathbf{x}) = \mathbf{c}(\mathbf{x}) = (c_e(x_e))_{e \in E}.$$

Theorem 1.3 *A flow $\overline{\mathbf{x}} \in X_{\mathbf{d}}$ is Nash if and only if $\overline{\mathbf{x}}$ is a minimizer of N on $X_{\mathbf{d}}$. In particular, Nash flows exist.*

Proof. By Theorem 1.2, $\overline{\mathbf{x}} \in X_{\mathbf{d}}$ is a minimizer of N if and only if

$$\nabla N(\overline{\mathbf{x}})(\mathbf{x} - \overline{\mathbf{x}}) = \overline{\mathbf{c}}(\mathbf{x} - \overline{\mathbf{x}}) \geq 0 \qquad \forall \mathbf{x} \in X_{\mathbf{d}}$$

where $\overline{c}_e = c_e(\overline{x}_e)$, $e \in E$. This amounts to saying that $\overline{\mathbf{x}}$ is cost optimal relative to the constant edge costs $\overline{c}_e$. So the claim follows from Theorem 1.1, observing that the (continuous) function N must achieve its minimum on the compact set $X_{\mathbf{d}}$ in some point $\overline{\mathbf{x}} \in X_{\mathbf{d}}$. $\square$

Ex. 1.3 *Show that $N(\mathbf{x})$ is convex. (Hint: Recall that a differentiable function $f : \mathbb{R} \to \mathbb{R}$ is convex if and only if f' is non-decreasing.)*

Ex. 1.4 *Show that $\widetilde{c}_e(x) = \frac{1}{x}\int_0^x c_e(t)dt$ is convex. Moreover, $\overline{\mathbf{x}} \in X_{\mathbf{d}}$ is Nash relative to the cost functions $c_e(x)$ iff $\overline{\mathbf{x}}$ is cost minimal relative to the edge costs $\widetilde{c}_e(x)$.*

Corollary 1.1 *Acyclic Nash flows always exist.*

Proof. Assume $\overline{\mathbf{x}} = \sum_i \overline{\mathbf{x}}^{(i)}$ is Nash, but not acyclic, *i.e.*, there exists a directed circuit $C_1 \subseteq E$ with $\overline{\mathbf{x}}^{(i)} > 0$ on C_1 for some i. Let

$$\varepsilon_1 := \min_{e \in C_1} \overline{x}_e^{(i)} > 0.$$

Then $\overline{\mathbf{x}}^{(i)} - \varepsilon_1 C_1$ is an $r_i - s_i$ flow of value d_i. Continuing this way, "removing" directed circuits from $\overline{\mathbf{x}}^{(i)}$, one at a time, we eventually end up with a acyclic (and hence a traffic) flow $\overline{\mathbf{x}}^{(i)} - \sum_k \varepsilon_k C_k$. Thus $\widetilde{\mathbf{x}} = \overline{\mathbf{x}} - \sum \varepsilon_k C_k \in X_{\mathbf{d}}$, and since $\widetilde{\mathbf{x}} < \overline{\mathbf{x}}$, we also have $N(\widetilde{\mathbf{x}}) \leq N(\overline{\mathbf{x}})$ by definition of N. So $\widetilde{\mathbf{x}}$ must be Nash as well. This shows that an acyclic Nash flow can be constructed from $\overline{\mathbf{x}}$ by simply removing directed circuits in finitely many steps. (Note that each time we remove a directed circuit some nonzero component of some $\overline{\mathbf{x}}^{(i)}$ is decreased to zero.) $\square$

Nash flows are not necessarily unique. Yet, the edge costs $\overline{c}_e = c_e(\overline{x}_e)$ of a Nash flow turn out to be unique and independent of the Nash flow $\overline{\mathbf{x}}$. As a consequence of this, also the Nash cost $C(\overline{\mathbf{x}})$ is unique:

Theorem 1.4 *Any two Nash flows $\overline{\mathbf{x}}, \overline{\overline{\mathbf{x}}} \in X_{\mathbf{d}}$ induce identical edge costs $c_e(\overline{x}_e) = c_e(\overline{\overline{x}}_e)$, $e \in E$, and identical Nash costs $C(\overline{\mathbf{x}}) = C(\overline{\overline{\mathbf{x}}})$.*

Proof. Let $\overline{c}_e = c_e(\overline{x}_e)$ and $\overline{\overline{c}}_e = c_e(\overline{\overline{x}}_e)$. As $c_e(x)$ is non-decreasing, we have $(\overline{c}_e - \overline{\overline{c}}_e)(\overline{x}_e - \overline{\overline{x}}_e) \geq 0$ for each $e \in E$. Hence

$$(\overline{\mathbf{c}} - \overline{\overline{\mathbf{c}}})^T(\overline{\mathbf{x}} - \overline{\overline{\mathbf{x}}}) = \sum_{e \in E} (\overline{c}_e - \overline{\overline{c}}_e)(\overline{x}_e - \overline{\overline{x}}_e) \geq 0.$$

Equality can hold only if $\overline{c}_e = \overline{\overline{c}}$ for each $e \in E$. (Note that $\overline{x}_e = \overline{\overline{x}}_e$ implies $\overline{c}_e = \overline{\overline{c}}_e$.) Hence, to establish the first claim, it suffices to show that $(\overline{\mathbf{c}} - \overline{\overline{\mathbf{c}}})^T(\overline{\mathbf{x}} - \overline{\overline{\mathbf{x}}}) \leq 0$. But this follows directly from the Nash property: Since $\overline{\mathbf{x}}$ is Nash, we have (*cf.* Theorem 1.1) $\overline{\mathbf{c}}^T(\overline{\mathbf{x}} - \overline{\overline{\mathbf{x}}}) \leq 0$ and, similarly, $\overline{\overline{\mathbf{c}}}^T(\overline{\overline{\mathbf{x}}} - \overline{\mathbf{x}}) \leq 0$. Hence

$$(\overline{\mathbf{c}} - \overline{\overline{\mathbf{c}}})^T(\overline{\mathbf{x}} - \overline{\overline{\mathbf{x}}}) = \overline{\mathbf{c}}(\overline{\mathbf{x}} - \overline{\overline{\mathbf{x}}}) + \overline{\overline{\mathbf{c}}}(\overline{\overline{\mathbf{x}}} - \overline{\mathbf{x}}) \leq 0.$$

The second claim follows from the fact that both $\overline{\mathbf{x}}$ and $\overline{\overline{\mathbf{x}}}$ are min cost flows relative to the same cost function $\overline{\mathbf{c}} = \overline{\overline{\mathbf{c}}}$, so $\overline{\mathbf{c}}^T\overline{\mathbf{x}} = \overline{\overline{\mathbf{c}}}^T\overline{\overline{\mathbf{x}}}$ must hold. □

1.2 Nash Flows versus min cost flows

In this section we answer the question "how bad" selfish routing can be, *i.e.*, how large the Nash cost $C(\overline{\mathbf{x}})$ can be compared to the minimum cost $C(\mathbf{x}^*)$. Basically, it turns out that the answer to this question depends only on the kind of cost functions we allow rather than on the network structure. For example in the case of linear cost functions $c_e(x) = \alpha_e x + \beta_e$, the worst case, *i.e.*, the minimum ration $C(\mathbf{x}^*)/C(\overline{\mathbf{x}})$ can be shown to be $3/4$, which is already attained in Pigou's two link network (*cf.* Figure 1.1). Similarly, if we allow polynomial cost functions of degree at most p, then, again the worst case ratio $C(\mathbf{x}^*)/C(\overline{\mathbf{x}})$ is already attained in Pigou's two link network as in Figure 1.2.

We first analyze the case of linear cost functions $c_e(x) = \alpha_e x + \beta_e$ ($\alpha_e \geq 0$, $\beta_e \geq 0$), as this case is particularly simple. First observe that

$$\nabla N(\mathbf{x}) = \mathbf{c}(\mathbf{x})^T = (\alpha_e x_e + \beta_e)_{e \in E}$$

and

$$\nabla C(\mathbf{x}) = (2\alpha_e x_e + \beta_e)_{e \in E}.$$

So $\nabla N(\mathbf{x}) = \nabla C(\frac{\mathbf{x}}{2})$ holds. This remarkable (*cf.* also Ex 1.5 below) relation is characteristic for the linear case and leads to a fairly simple proof of

Theorem 1.5 *In the case of linear cost functions, the minimum ration $C(\mathbf{x}^*)/C(\overline{\mathbf{x}})$ (min cost to Nash cost) is at least 3/4 in any network.*

Proof. Convexity of C yields

$$\begin{aligned} C(\mathbf{x}^*) &\geq C(\frac{\overline{\mathbf{x}}}{2}) + \nabla C(\frac{\overline{\mathbf{x}}}{2})(\mathbf{x}^* - \frac{\overline{\mathbf{x}}}{2}) \\ &= C(\frac{\overline{\mathbf{x}}}{2}) + \nabla N(\overline{\mathbf{x}})(\mathbf{x}^* - \frac{\overline{\mathbf{x}}}{2}) \\ &= C(\frac{\overline{\mathbf{x}}}{2}) + \nabla N(\overline{\mathbf{x}})(\mathbf{x}^* - \overline{\mathbf{x}}) + \nabla N(\overline{\mathbf{x}})\frac{\overline{\mathbf{x}}}{2} \\ &\geq C(\frac{\overline{\mathbf{x}}}{2}) + \frac{1}{2}C(\overline{\mathbf{x}}), \end{aligned}$$

due to the Nash condition $\nabla N(\overline{\mathbf{x}})(\mathbf{x} - \overline{\mathbf{x}}) \geq 0$ and $\nabla N(\mathbf{x}) = \mathbf{c}(\mathbf{x})$.

We are left to check that $C(\frac{\mathbf{x}}{2}) \geq \frac{1}{4}C(\mathbf{x})$:

$$C(\frac{\mathbf{x}}{2}) = \sum_{e \in E}(\alpha_e \frac{x_e}{2} + \beta_\varepsilon)\frac{x_e}{2} \geq \sum_{e \in E}(\alpha_e \frac{x_e}{2} + \frac{\beta_e}{2})\frac{x_e}{2} = \frac{1}{4}C(\mathbf{x}).$$

□

Ex. 1.5 *Show that (in case of linear edge costs)* $\overline{\mathbf{x}} \in X_{\mathbf{d}}$ *is Nash if and only if* $\overline{\mathbf{x}}/2 \in X_{\mathbf{d}/2}$ *is cost minimal.*

Non-linear edge costs. In the following we present a result of Roughgarden (2002) stating that also for non-linear edge costs $c_e(x)$, Pigou's network already provides the worst case (in a sense to be specified below). We need to restrict ourselves, however, to cost functions $c_e(x)$ that are *standard* in the sense that $c_e(x)$ is differentiable and $c_e(x) \cdot x$ is convex. (The latter amounts to say that the cost function $C(\mathbf{x})$ is convex on $\mathbb{R}^E_+$, so that computing a min cost flow is a convex problem.) This assumption is not very restrictive. Indeed quite often even the cost functions $c_e(x)$ itself are convex.

Ex. 1.6 *Show (in the case of standard cost functions) that* $\overline{\mathbf{x}} \in X_{\mathbf{d}}$ *is cost minimal relative to the edge costs* $c_e(x)$ *iff* $\overline{\mathbf{x}}$ *is Nash relative to the edge costs* $\widetilde{c}_e(x) = c'_e(x) + c_e(x)$.

Let $c(x)$ be a standard cost function. Relative to a given demand $d \geq 0$ we define a corresponding *Pigou network* for $c(x)$ to be a network consisting ot two links from r to s, one with cost function $c(x)$, and the other with constant cost $c(d)$.

The Nash flow $\overline{\mathbf{x}}$ in the Pigou network for $c(x)$ and demand $d \geq 0$ routes all the demand along the link with cost $c(x)$, so the Nash cost is $C(\overline{\mathbf{x}}) = dc(d)$. The min

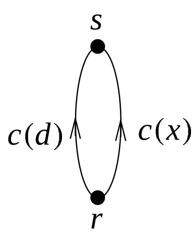


Figure 1.4: Pigou network for c and demand $d \geq 0$.

cost flow x^* routes a certain amount λd along the $c(x)$ cost link and $(1-\lambda)d$ along the constant cost link. So

$$C(\mathbf{x}^*) = \min_{\lambda \in [0,1]} (1-\lambda)dc(d) + \lambda dc(\lambda d).$$

The Pigou ratio $\rho(c)$ is defined to be the minimum ratio $C(\mathbf{x}^*)/C(\overline{\mathbf{x}})$, where the minimum is taken over all Pigou networks for c, *i.e.*, over all demands $d \geq 0$:

$$\rho(c) = \min_{d \geq 0} \min_{\lambda \in [0,1]} 1 - \lambda + \lambda c(\lambda d)/c(d).$$

(In case $c(d) = 0$, *i.e.*, when the Nash cost $C(\overline{x})$ is equal to 0, we interprete $c(\lambda d)/c(d)$ as 1.)

For a concrete cost function $c(x)$, the Pigou ratio $\rho(c)$ is usually easy to compute. Note that for fixed $d \geq 0$, the inner minimization problem is convex and can be solved by differentiation.

Ex. 1.7 *Let $c(x) = x^k$. Then*

$$\begin{aligned} \rho(c) &= \min_{d \geq 0} \min_{\lambda \in [0,1]} 1 - \lambda + \lambda^{k+1} \\ &= \min_{\lambda \in [0,1]} 1 - \lambda + \lambda^{k+1}. \end{aligned}$$

The minimum is achieved in $\lambda = \sqrt[k]{1/(k+1)}$, yielding a Pigou ratio $\rho(c) = 1 - \sqrt[k]{1/(k+1)}\ k/(k+1)$, as in the nonlinear variant of Pigon's network, cf. *Figure 1.2.*

Theorem 1.6 *In a traffic network with standard edge cost functions $c_e(x)$, $e \in E$, the min cost to Nash cost ratio can be bounded from below as follows:*

$$C(\mathbf{x}^*)/C(\overline{\mathbf{x}}) \geq \min_{e \in E} \rho(c_e).$$

Proof. Let us introduce for the moment the following notation: For $\mathbf{x} \in \mathbb{R}^E$ and $\boldsymbol{\lambda} \in [0,1]^E$ we define $\boldsymbol{\lambda}\mathbf{x} \in \mathbb{R}^E$ to be the vector with components $\lambda_e x_e$, $e \in E$.

Now let $\overline{\mathbf{x}} \in X_{\mathbf{d}}$ be Nash and $\boldsymbol{\lambda} \in [0,1]^E$ (to be determined below). Convexity of $C(\mathbf{x})$ yields

$$C(\mathbf{x}^*) \geq C(\boldsymbol{\lambda}\overline{\mathbf{x}}) + \nabla C(\boldsymbol{\lambda}\overline{\mathbf{x}})(\mathbf{x}^* - \boldsymbol{\lambda}\overline{\mathbf{x}}). \qquad (*)$$

The e^{th} component of $\nabla C(\boldsymbol{\lambda}\overline{\mathbf{x}})$ is

$$(\nabla C(\boldsymbol{\lambda}\overline{\mathbf{x}}))_e = c_e(\lambda_e \overline{x}_e) + c_e'(\lambda_e \overline{x}_e)\lambda_e \overline{x}_e.$$

If $\lambda_e = 0$ resp. $\lambda_e = 1$, the e^{th} component of $\nabla C(\boldsymbol{\lambda}\overline{\mathbf{x}})$ equals $c_e(0) \leq c_e(\overline{x}_e)$ resp. $c_e(\overline{x}_e) + c_e'(\overline{x}_e)\overline{x}_e \geq c_e(\overline{x}_e)$. So we can find $\boldsymbol{\lambda} \in [0,1]^E$ such that

$$\nabla C(\boldsymbol{\lambda}\overline{\mathbf{x}}) = \big(c_e(\overline{x}_e)\big)_{e \in E} \quad \big[= \mathbf{c}(\overline{\mathbf{x}})\big].$$

Substituting $\nabla C(\boldsymbol{\lambda}\overline{\mathbf{x}}) = \mathbf{c}(\overline{\mathbf{x}})$ in $(*)$ yields

$$\begin{aligned} C(\mathbf{x}^*) &\geq C(\boldsymbol{\lambda}\overline{\mathbf{x}}) + \mathbf{c}(\overline{\mathbf{x}})^T(\mathbf{x}^* - \boldsymbol{\lambda}\overline{\mathbf{x}}) \\ &\geq C(\boldsymbol{\lambda}\overline{\mathbf{x}}) + \mathbf{c}(\overline{\mathbf{x}})^T(\overline{\mathbf{x}} - \boldsymbol{\lambda}\overline{\mathbf{x}}), \end{aligned}$$

since $\overline{\mathbf{x}}$ is cost optimal relative to $\overline{\mathbf{c}} = \mathbf{c}(\overline{\mathbf{x}})$. So

$$\begin{aligned} C(\mathbf{x}^*) &\geq \sum_{e \in E} c_e(\lambda_e \overline{x}_e)\lambda_e \overline{x}_e + (1-\lambda_e)c_e(\overline{x}_e)\overline{x}_e \\ &= \sum_{e \in E} \left[\frac{c_e(\lambda_e \overline{x}_e)}{c_e(\overline{x}_e)}\lambda_e + 1 - \lambda_e\right] c_e(\overline{x}_e)\overline{x}_e \\ &\geq \sum_{e \in E} \rho(c_e)c_e(\overline{x}_e)\overline{x}_e \geq \min_{e \in E} \rho(c_e) C(\overline{\mathbf{x}}). \end{aligned}$$

$\square$

Ex. 1.8 *Show that if all edge costs c_e, $e \in E$ are polynomials of degree at most k, then the ration $C(\mathbf{x}^*)/C(\overline{\mathbf{x}})$ is bounded from below by $\rho(x^k)$. (Hint: Replace an edge $e \in E$ with cost $c_e(x) = \alpha_k x^k + \ldots + \alpha_0$ by a path of length $k+1$ with edge costs $\alpha_k x^k, \ldots, \alpha_0$ resp.)*

Ex. 1.9 *Derive Theorem 1.5 from Theorem 1.6.*

1.3 Monotonicity properties

Here we assume for simplicity that each edge cost $c_e(x)$ is strictly increasing. As a consequence, $N(\mathbf{x})$ is strictly convex and hence, for each demand $\mathbf{d} \geq 0$ there is a unique Nash flow $\overline{\mathbf{x}} = \overline{\mathbf{x}}(\mathbf{d})$. We want to analyse $\overline{\mathbf{x}} = \overline{\mathbf{x}}(\mathbf{d})$ as a function of $\mathbf{d}$.

Theorem 1.7 *The Nash flow $\overline{\mathbf{x}} = \overline{\mathbf{x}}(\mathbf{d})$ is a continuous function of* $\mathbf{d}$.

Proof. Let $\mathbf{d} \geq \mathbf{0}$ and $\mathbf{d}_k \to \mathbf{d}$. We are to show that $\overline{\mathbf{x}}(\mathbf{d}_k) \to \overline{\mathbf{x}}(\mathbf{d})$. Assume w.l.o.g. that $\|\mathbf{d}_k - \mathbf{d}\| \leq 1$ for all k. Then

$$X = \bigcup_{\mathbf{d}':\|\mathbf{d}-\mathbf{d}'\|\leq 1} X_{\mathbf{d}'}$$

is compact and hence $\overline{\mathbf{x}}(\mathbf{d}_k)$ contains a converging subsequence. We are to show that each such converging subsequence converges to $\overline{\mathbf{x}}(\mathbf{d})$.

Fix a converging subsequence of $\overline{\mathbf{x}}(\mathbf{d}_k)$. For notational convenience, assume w.l.o.g. that $\overline{\mathbf{x}}(\mathbf{d}_k)$ itself is convergent. For each $\mathbf{d}_k$, we can construct a flow $\mathbf{x}_k \in X_{\mathbf{d}_k}$ from $\overline{\mathbf{x}}(\mathbf{d})$ by (small) flow augmentations resp. reductions on the various O-D pairs (r_i, s_i) (depending on whether the i^{th} component of $\mathbf{d}_k$ is larger or smaller than the i^{th} component of $\mathbf{d}$). These flows $\mathbf{x}_k \in C_{\mathbf{d}_k}$ then converge to $\overline{\mathbf{x}}(\mathbf{d})$. From the Nash property of the flows $\overline{\mathbf{x}}(\mathbf{d}_k)$ we conclude that

$$N(\overline{\mathbf{x}}(\mathbf{d}_k)) \leq N(\mathbf{x}_k) \qquad \text{for all } k.$$

Taking the limit $k \to \infty$, we thus obtain

$$N(\lim_{k\to\infty} \overline{\mathbf{x}}(\mathbf{d}_k)) \leq N(\overline{\mathbf{x}}(\mathbf{d})),$$

and since $N(\overline{\mathbf{x}}(\mathbf{d}))$ is the unique minimizer of N on $X_{\mathbf{d}}$, actually $\lim_{k\to\infty} \overline{\mathbf{x}}(\mathbf{d}_k) = \overline{\mathbf{x}}(\mathbf{d})$ must hold. $\square$

We next ask the question how $\overline{\mathbf{x}}(\mathbf{d})$ changes "qualitatively" if we vary the demand vector $\mathbf{d} \geq \mathbf{0}$. Intuitively, we would expect that increasing the demand d_i would increase the "travel time" from r_i to s_i (*i.e.* the length of a min cost path from r_i to s_i) in the Nash equilibrium. This is indeed true (*cf.* Theorem 1.8 below).

For simplicity, we restrict ourselves to the case of a single O-D pair (r, s). For a given demand $d \geq 0$, define the node potentials $\overline{y}_j$ to be the length of a min cost path from r to j relative to the edge costs $\overline{\mathbf{c}}$ induced by $\overline{\mathbf{x}}$. We claim that $\overline{y}_s$ increases with d. First note that

$$\overline{y}_k \leq \overline{y}_j + \overline{c}_e, \qquad e = (j, k) \in E$$

holds by definition of the node potentials $\overline{y}_j$. Moreover, equality must hold whenever $\overline{x}_e > 0$. (Recall that $\overline{\mathbf{x}}$ routes all the demand along min cost paths from r to s.)

Now assume that, relative to a larger demand $\widetilde{d} > d$ we have a Nash flow $\widetilde{\mathbf{x}}$ with associated edge costs $\widetilde{\mathbf{c}}$ and corresponding node potentials $\widetilde{y}_j$. We are to show that $\widetilde{y}_s \geq \overline{y}_s$ holds. To prove this, first observe that $\widetilde{\mathbf{x}}$ is an $r - s$ flow of value $\widetilde{d} > d$, so there must be an augmenting path relative to $\overline{\mathbf{x}}$, *i.e.*, an $r - s$ path P such that

$$\widetilde{\mathbf{x}} > \overline{\mathbf{x}} \text{ on } P^+ \quad \text{and} \quad \widetilde{\mathbf{x}} < \overline{\mathbf{x}} \text{ on } P^-.$$

We show that $\widetilde{y}_j \geq \overline{y}_j$ holds for all nodes on P. This is certainly true for $j = r$ (as $\widetilde{y}_r = \overline{y}_r = 0$.) We proceed by induction along the path P. Let j be some node on P and let k be the next node on P (in direction towards s). If $e = (j, k)$ is a forward edge ($e \in P^+$), then $\widetilde{x}_e > \overline{x}_e$ implies both $\widetilde{c}_e \geq \overline{c}_e$ and $\widetilde{x}_e > 0$. This yields

$$\widetilde{y}_k = \widetilde{y}_j + \widetilde{c}_e \geq \overline{y}_j + \overline{c}_e \geq \overline{y}_k.$$

Similarly, if $e = (k, j)$ is a backward edge, then $\overline{x}_e > \widetilde{x}_e$ implies $\overline{x}_e > 0$ and $\overline{c}_e \geq \widetilde{c}_e$. Hence

$$\overline{y}_k = \overline{y}_j - \overline{c}_e \leq \widetilde{y}_j - \widetilde{c}_e \leq \widetilde{y}_k,$$

completing the inductive proof.

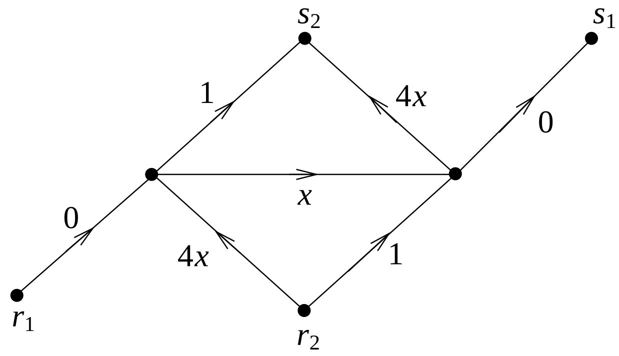
As mentioned above, this result can be generalized to the case of multiple O-D pairs as follows:

Theorem 1.8 *The travel time from r_i to s_i at Nash equlibrium is a non-decreasing function of d_i (assuming all other components of* **d** *are fixed).*

□

We like to stress that nothing can be said in general about the travel times between other O-D pairs. Indeed it may happen that increasing d_i may result in a *decrease* of the travel times between other O-D pairs (r_k, s_k), $k \neq i$ (*cf.* Ex 1.10 below).

Ex. 1.10 *Consider the network below with two O-D pairs and cost functions as indicated. Compute a Nash flow* $\overline{\mathbf{x}}$ *for* $\mathbf{d} = (0, 1/5)$ *and explain what happens if* d_1 *or* d_2 *is increased.*



Ex. 1.11 *Prove or disprove: In a single O-D pair network, replacing an edge cost function* $c_e(x)$ *by* $\widetilde{c}_e(x) \geq c_e(x)$ *may result in a decrease of the travel time.*